Notes on the Second LP Bound and Association Schemes

Written by Chris Jones *February 2026*

1 Delsarte LPs for Codes

We say that $C \subseteq \mathbb{F}_2^n$ is a distance-d code if $|x - y| \geq d$ for all $x, y \in C$ with $x \neq y$. What is the largest possible size of a distance-d code in $\mathbb{F}_2^n$? Call this $A(n, d)$.

Here is an integer program expressing $A(n, d)$:

	Variables:	$f : \mathbb{F}_2^n \to \{0, 1\}$	
max	$\sum_{x \in \mathbb{F}_2^n} f(x)$		
s.t.	$f(x)f(y) = 0$	if $\|x - y\| \leq d$ and $x \neq y$	(Distance constraints)

All terms have degree at most 2 in f, so this program admits a natural SDP relaxation. Moreover, if we recognize this as Maximum Independent Set in the graph with vertex set $\mathbb{F}_2^n$ and edges between vertices at distance $\leq d$, we can remember that the SDP relaxation of Independent Set is the *Lovász ϑ' function*. This SDP is written below, and it is not hard to see that it is a relaxation of the integer program.[1]

	Variables:	$\mathbf{M} \in \mathbb{R}^{\mathbb{F}_2^n \times \mathbb{F}_2^n}$ symmetric	
max	$\sum_{x, y \in \mathbb{F}_2^n} \mathbf{M}[x, y]$		
s.t.	$\mathrm{tr}(\mathbf{M}) = 1$		(Normalization)
	$\mathbf{M}[x, y] = 0$	if $\|x - y\| \leq d$ and $x \neq y$	(Distance constraints)
	$\mathbf{M}[x, y] \geq 0$		(Non-negativity)
	$\mathbf{M} \succeq 0$		(PSDness)

Dual of the ϑ' SDP:

	Variables:	$\mathbf{N} \in \mathbb{R}^{\mathbb{F}_2^n \times \mathbb{F}_2^n}$ symmetric	
min	$\mathrm{tr}(\mathbf{N})$		
s.t.	$\sum_{x, y \in \mathbb{F}_2^n} \mathbf{N}[x, y] = 2^n$		(Normalization)
	$\mathbf{N}[x, y] \leq 0$	if $\|x - y\| \geq d$	(Distance constraints)
	$\mathbf{N} \succeq 0$		(PSDness)

Therefore, to bound the value of $A(n, d)$, we construct feasible solutions to the dual SDP.

[1] There is a small difference between the ϑ function and the ϑ' function, which is that the ϑ' function has additional non-negativity constraints $\mathbf{M}[x, y] \geq 0$. The ϑ function is also equal to the degree-2 Sum-of-Squares relaxation of the integer program.

1.1 Proof of the second LP bound

Theorem 1.1 (Second LP bound [MRRW77]). *Let $d = \delta n$ with $\delta \in (0, \frac{1}{2})$. Then*

$$\frac{1}{n}\log_2 A(n,d) \le \min_{\delta/2 \le \alpha \le 1/2} 1 - h_2(\alpha) + h_2(\tau)$$

where h_2 is the binary entropy function and $\tau = \tau(\alpha, \delta)$ is defined implicitly by the equation

$$\delta = \frac{(\alpha - \tau)(1 - \alpha - \tau)}{1 + 2\sqrt{\tau(1-\tau)}}.$$

The idea is to construct the dual solution $\mathbf{N}$ inside an *association scheme*, specifically the *Johnson scheme* (the simpler *Hamming scheme* is used for the weaker first LP bound). We follow the conceptual approach of Linial-Loyfer [LL23] and Barg-Nogin [BN06] based on the theory of association schemes included in Section 2.

Fix $k \in \{0, 1, \ldots, n/2\}$ and let $\mathbf{A}_0, A_2, A_4, \ldots, \mathbf{A}_{2k} \in \mathbb{R}^{\mathbb{F}_2^n \times \mathbb{F}_2^n}$ be the matrices of the Johnson scheme $J(n,k)$,

$$\mathbf{A}_i[x,y] = \begin{cases} 1 & |x| = k, |y| = k, |x-y| = i \\ 0 & \text{otherwise} \end{cases}$$

The candidate solution is:

$$\mathbf{N} = \sum_{i=0}^{k} g(i)\mathbf{A}_{2i}$$

for $g : \{0, 1, \ldots, k\} \to \mathbb{R}$. Let $Q_t(x), t \in \{0, 1, \ldots, k\}$ be the Q-polynomials of the Johnson scheme. We choose $g(x) = (Q_1(x)^m - Q_1(d)^m)Q_t(x)^2$ for an odd $m \in \mathbb{N}$ and some choice of $t \in \{0, 1, \ldots, k\}$ to be decided later (both k and t will be $\Theta(n)$ while $\log n \ll m \ll \sqrt{n}$. The solution is not yet normalized, either).

Observe that $\mathbf{N}$ satisfies the distance constraints since $Q_t(x)^2 \ge 0$ and Q_1 is a linear function [Del73]:[2]

$$Q_1(x) = \frac{(n-1)}{2k(n-k)}\left(2k(n-k) - nx\right)$$

The crucial PSDness constraint for $\mathbf{N}$ reduces to checking that the coefficients of g are non-negative in the basis of Q-functions (Proposition 2.8). We will choose m, t so that $(Q_1^m - Q_1(d)^m)Q_t$ has a non-negative Q-function expansion. Then multiplying with the remaining factor of Q_t implies that the expansion of g is still non-negative since the "Krein parameters" satisfy $q_{ij}^k \ge 0$ (Proposition 2.15).

In the expansion of $(Q_1^m - Q_1(d)^m)Q_t = Q_1^m Q_t - Q_1(d)^m Q_t$ we can see that the first term is a non-negative combination of Q-polynomials (again since $q_{ij}^k \ge 0$) whereas the second term is the polynomial Q_t with a negative coefficient $-Q_1(d)^m$. Therefore, all of the coefficients

[2] To make our proofs self-contained, it would suffice to calculate the function μ which counts the dimensions of the eigenspaces of the Johnson scheme, given by $\mu(t) = \binom{n}{t} - \binom{n}{t-1}$. The Q-polynomials are the orthogonal polynomials for the simpler measure ν (Proposition 2.12) with normalization given by μ (Proposition 2.12).

on $Q_s, s \neq t$ are guaranteed to be non-negative and we just need to ensure that the coefficient on Q_t remains non-negative.

To compute $Q_1^m Q_t$ we repeatedly use the three-term recurrence for the Q-polynomials:

$$Q_1 Q_s = aQ_{s-1} + bQ_s + cQ_{s+1} \tag{1}$$

for some coefficients $a, b, c \geq 0$ which depend on s. For the Johnson scheme, we have the following explicit formula [Del73, BN06]:[3]

$$\begin{aligned} Q_1 Q_t =& \frac{n(n-1)(t+1)(k-t)(n-k-t)}{2k(n-k)(n-2t)(n-2t+1)} \cdot Q_{t-1} \\ &+ \left(n - 1 - \frac{n(n-1)\big((n+2)k(n-k) - nt(n-t+1)\big)}{2k(n-k)(n-2t)(n-2t+2)} \right) \cdot Q_t \\ &+ \frac{n(n-1)(k-t+1)(n-k-t+1)(n-t+2)}{2k(n-k)(n-2t+2)(n-2t+3)} \cdot Q_{t+1} \,. \end{aligned} \tag{2}$$

We can make simplifications since k, t, n are on the scale $\Theta(n)$:

$$\begin{aligned} Q_1 Q_t &\approx \frac{n^2 t(k-t)(n-k-t)}{2k(n-k)(n-2t)^2} \cdot Q_{t-1} \\ &\quad + \left(n - \frac{n^3(k-t)(n-k-t)}{2k(n-k)(n-2t)^2} \right) \cdot Q_t \\ &\quad + \frac{n^2(k-t)(n-k-t)(n-t)}{2k(n-k)(n-2t)^2} \cdot Q_{t+1} \end{aligned} \tag{3}$$

$$\begin{aligned} &= n \cdot Q_t + \frac{n(k-t)(n-k-t)}{2k(n-k)(n-2t)^2} \left(nt \cdot Q_{t-1} - n^2 \cdot Q_t + n(n-t) \cdot Q_{t+1} \right) \\ &=: a' Q_{t-1} + b' Q_t + c' Q_{t+1} \,. \end{aligned} \tag{4}$$

In summary, to calculate $Q_1^m Q_t$, we interpret Eq. (1) as taking a walk on $\{0, 1, \ldots, k\}$ which starts at t and moves at each step from s to either $s-1, s$, or $s+1$. The final coefficient on Q_t can be computed by summing over walks which return to t.

Let $\doteq$ denote equality up to polynomial factors: $x \doteq y$ if $\frac{1}{p(n)} \cdot y \leq x \leq q(n) \cdot y$ for some $p, q \leq O(n^c)$ for a constant c.

Lemma 1.2. *The coefficient of Q_t in $Q_1^m Q_t$ is $\doteq$ to $(b' + 2\sqrt{a'c'})^m$.*

Proof. The coefficient of a given walk is the product of a, b, c in Eq. (1) for each step of the walk. Convert the approximation in Eq. (3) into $a = a' \cdot (1 \pm O(1/n))$ and similar for b, c. Then the coefficient of a given walk is $(a')^p (b')^q (c')^r \cdot (1 \pm O(m/n))$ where p, q, r are the number of $-1, 0$, and $+1$ steps respectively.

[3]The three-term recurrence for a family of orthogonal polynomials has an explicit form based only on μ and ν. Maybe there is also a combinatorial proof of this formula using the Johnson scheme eigenspaces?

In order for the walk to return to t, the number of $+1$ steps and -1 steps must be the same. Let this number be p. The number of walks with p such steps is $\binom{m}{p,p,m-2p}$ and the coefficient of a given walk is $(a'c')^p(b')^{m-2p}\cdot(1\pm O(m/n))$. Therefore, the total coefficient is

$$\begin{aligned}&\sum_{p=0}^{m/2}\binom{m}{p,p,m-2p}(a'c')^p(b')^{m-2p}\cdot(1\pm O(m/n))\\ =&\left(\sum_{p=0}^{m/2}\binom{m}{p,p,m-2p}(a'c')^p(b')^{m-2p}\right)\cdot(1\pm O(m^2/n))\end{aligned}$$

Estimate the interior summation:

$$\begin{aligned}&\sum_{p=0}^{m/2}\binom{m}{p,p,m-2p}(a'c')^p(b')^{m-2p}=\sum_{p=0}^{m/2}\binom{m}{2p}\binom{2p}{p}(a'c')^p(b')^{m-2p}\\ &\doteq\sum_{p=0}^{m/2}\binom{m}{2p}4^p(a'c')^p(b')^{m-2p}\doteq(b'+2\sqrt{a'c'})^m\,.\end{aligned}$$

□

Recall that we want Lemma 1.2 to be larger than $Q_1(d)^m$. We have

$$Q_1(d)=\frac{(n-1)}{2k(n-k)}\left(2k(n-k)-nd\right)=n-\frac{n^2d}{2k(n-k)}+O(1)$$

Comparing the base of the exponent in Lemma 1.2 against $Q_1(d)$:

$$\begin{aligned}&b'+2\sqrt{a'c'}>n-\frac{n^2d}{2k(n-k)}\\ \iff& n(k-t)(n-k-t)(-n^2+2n\sqrt{t(n-t)})>-n^2d(n-2t)^2\\ \iff& d>\frac{(k-t)(n-k-t)(n-2\sqrt{t(n-t)})}{(n-2t)^2}\\ \iff& d>\frac{(k-t)(n-k-t)}{n+2\sqrt{t(n-t)}}\,.\end{aligned}$$

Divide by n and define the limiting constant parameters $\delta=\frac{d}{n},\tau=\frac{t}{n},\alpha=\frac{k}{n}$,

$$\iff \delta>\frac{(\alpha-\tau)(1-\alpha-\tau)}{1+2\sqrt{\tau(1-\tau)}} \tag{5}$$

In conclusion, if Eq. (5) holds, then the base of the exponent in Lemma 1.2 is larger than that of $Q_1(d)$ by a constant factor. Taking $m\gg\log n$ to control the polynomial error term in Lemma 1.2, we obtain a positive lower bound of $n^{\Omega(m)}$ on the coefficient on Q_t and we have satisfied the PSDness constraint.

Finally we calculate the value of the candidate solution. Normalizing the solution and calculating on the exponential scale, we show

$$\frac{1}{n}\log_2\left(\frac{2^n\,\mathrm{tr}(\mathbf{N})}{\sum_{x,y\in\mathbb{F}_2^n}\mathbf{N}[x,y]}\right) \le 1 - h_2(\alpha) + h_2(\tau) + o(1)\,. \tag{6}$$

The trace is

$$\mathrm{tr}(\mathbf{N}) = \binom{n}{k}g(0) = \binom{n}{k}(Q_1(0)^m - Q_1(d)^m)Q_t(0)^2 = \binom{n}{k}(Q_1(0)^m - Q_1(d)^m)\left(\binom{n}{t} - \binom{n}{t-1}\right)^2 \tag{7}$$

The last equality is Proposition 2.24 using $\mu(t) = \binom{n}{t} - \binom{n}{t-1}$ for the Johnson scheme [Del73]. The denominator is

$$\sum_{x,y\in\mathbb{F}_2^n}\mathbf{N}[x,y] = \binom{n}{k}^2 \mathop{\mathbb{E}}_{x\sim\nu} g(x) \tag{8}$$

where ν is the distribution on $\{0,1,\dots,k\}$ such that $\nu(i) \propto \binom{k}{i}\binom{n-k}{i}$. According to the PSDness analysis, write $g(x) = \left(\sum_{s=0}^{t} c_s Q_s\right)Q_t$ where $c_s \ge 0$ and $c_t \ge n^{\Omega(m)} \ge 1$. The Q-polynomials are orthogonal under ν (Corollary 2.13), therefore,

$$\mathop{\mathbb{E}}_{x\sim\nu} g(x) = c_t \mathop{\mathbb{E}}_{x\sim\nu}[Q_t(x)^2] \ge \mathop{\mathbb{E}}_{x\sim\nu}[Q_t(x)^2] = \binom{n}{t} - \binom{n}{t-1}. \tag{9}$$

The last equality is Proposition 2.12. Combining Eqs. (8), (9) and (7) yields Eq. (6) after we ignore the term involving m (which is subexponential since $m \ll n$) and use the binomial coefficient estimate $\frac{1}{n}\log_2\binom{n}{\alpha n} = h_2(\alpha) + o(1)$.

Remark 1.3. *Eq. (5) is not feasible if $\alpha < \delta/2$. The explicit feasible range for τ is*

$$\alpha > \tau > \frac{1}{2} - \frac{1}{2}\sqrt{1 - \left(\delta - \sqrt{4\alpha(1-\alpha) - 2\delta + \delta^2}\right)^2}\,.$$

2 Association Schemes

2.1 Overview

Delsarte's theory of association schemes is a beautiful theory of the combinatorial and spectral properties of certain matrix families. There are several possible perspectives on this theory. We will emphasize the "analytic" perspective: an association scheme is a matrix algebra which admits a special type of spectral analysis, similar to Fourier analysis. The goal of these notes is to synthesize the key results of the theory so that they may be used for matrix analysis problems such as in Section 1.

In particular, we restrict ourselves to *Schurian* association schemes, which are algebras of matrices invariant under a group action.[4] Given a transitive group action of G on a finite set Ω, the corresponding Schurian association scheme consists of matrices $\mathbf{M} \in \mathbb{R}^{\Omega\times\Omega}$ which are G-invariant:

$$\forall g \in G.\ \forall x, y \in \Omega.\ \mathbf{M}[x,y] = \mathbf{M}[g\cdot x, g\cdot y]\,.$$

The *Hamming scheme* consists of matrices $\mathbf{M} \in \mathbb{R}^{\mathbb{F}_2^n\times\mathbb{F}_2^n}$ such that $\mathbf{M}[x,y]$ only depends on $|x-y|$. The *Johnson scheme* consists of matrices $\mathbf{M} \in \mathbb{R}^{J(k)\times J(k)}$ such that $\mathbf{M}[x,y]$ only depends on $|x-y|$ where $J(n,k) := J(k) := \{x \in \mathbb{F}_2^n : |x| = k\}$ is the *slice*. These are the Schurian association schemes for $\Omega = \mathbb{F}_2^n$ and $\Omega = J(k)$, respectively, with the action of S_n by permuting the bits.

A matrix in the scheme can be efficiently represented by only its distinct entries. Association scheme theory describes the equivalence between spectral analysis of the matrices in the scheme, and analysis of these compressed representations (which are exponentially smaller representations for the Hamming and Johnson schemes). Moreover, there is a duality theory for association schemes which gives the association scheme additional structure (a simple and nontrivial example is that the number of distinct eigenspaces equals the number of distinct matrix entries, Proposition 2.3).

Although we describe the theory of association schemes in terms of matrices, a completely equivalent viewpoint is through the analysis of *spherical functions*, such as (for the Johnson scheme) $f : J(k) \to \mathbb{R}$ which only depends on the Hamming distance to a fixed point $x_0 \in J(k)$. Some ideas also generalize to spherical functions on infinite domains, such as radial functions on $\mathbb{R}^n$ and spherical functions on S^{n-1}, the surface of the sphere.[5] *Gelfand pairs* provide a general framework for harmonic analysis in these settings, and at the end of the note, we prove the exact correspondence between finite Gelfand pairs and commutative Schurian association schemes.

The proofs themselves are rather trivial; almost none are more than a few lines long. Despite this, the overall analytical framework is very strong and should not be underestimated!

2.2 Matrix and function viewpoints

Definition 2.1 (Association scheme)**.** *For a finite set Ω and a partition $\mathcal{R}$ of $\Omega\times\Omega$, the pair $(\Omega,\mathcal{R})$ is an association scheme if:*

(i) one of the blocks of $\mathcal{R}$ equals $\{(x,x) : x \in \Omega\}$,

(ii) for all $r \in \mathcal{R}$, the transpose of r is also in $\mathcal{R}$,

[4] While Schurian association schemes are arguably the most common schemes encountered in practice (in particular the Hamming and Johnson schemes), the modern definition of an association scheme is purely combinatorial, so-called "group theory without groups" [BBIT21]. This allows to study e.g. distance-transitive graphs within the same framework. *Coherent configurations* are a further combinatorial generalization.

[5] The sphere packing problem in these spaces also admits analogous linear programming bounds (Cohn–Elkies bound for $\mathbb{R}^n$ and Kabatiansky–Levenshtein bound for S^{n-1}).

(iii) for all $r_i, r_j, r_k \in \mathcal{R}$ *there exists an integer* $p_{ij}^k \in \mathbb{N}$ *such that,*

$$\forall (x,z) \in r_k.\ |\{y \in \Omega : (x,y) \in r_i, (y,z) \in r_j\}| = p_{ij}^k\,.$$

The elements of $\mathcal{R}$ *are called the relations of the scheme and* p_{ij}^k *the intersection numbers.*

Each association scheme corresponds to a space of $\Omega \times \Omega$ matrices called the *Bose–Mesner algebra* of the scheme. A matrix in the Bose–Mesner algebra is said to be "in the scheme".

Definition 2.2 (Bose–Mesner algebra)**.** *The Bose–Mesner algebra of an association scheme* $(\Omega, \mathcal{R})$ *is the matrix algebra generated by the matrices* $\mathbf{M}_r \in \mathbb{R}^{\Omega\times\Omega}$ *for* $r \in \mathcal{R}$ *defined by*

$$\mathbf{M}_r[x,y] = \begin{cases} 1 & (x,y) \in \textit{relation } r \\ 0 & \textit{otherwise}\,. \end{cases}$$

The span of the $\mathbf{M}_r$ is closed under matrix multiplication by definition of the parameters p_{ij}^k in Definition 2.1. Hence the $\mathbf{M}_r$ are a basis for the Bose–Mesner algebra.

In order to make the core spectral theory work, we henceforth assume that the scheme is *symmetric* and *commutative* (i.e. the matrices $\mathbf{M}_r$ are symmetric and commute). We also study only *Schurian* schemes, defined by a finite set Ω and a group acting transitively on Ω, and letting the relations $\mathcal{R}$ be the orbits of G on $\Omega \times \Omega$.

For Schurian schemes, the Bose–Mesner algebra consists exactly of the set of G-invariant matrices $\mathbf{M} \in \mathbb{R}^{\Omega\times\Omega}$, which satisfy $\mathbf{M}[x,y] = \mathbf{M}[g\cdot x, g\cdot y]$ for all $x, y \in \Omega, g \in G$. Closure under matrix multiplication is also clear for these schemes since the product of two G-invariant matrices is G-invariant.

There are two equivalent viewpoints of the unsymmetrized objects in a scheme $(\Omega, \mathcal{R})$:

1. A matrix $\mathbf{M} \in \mathbb{R}^{\Omega\times\Omega}$.
2. A function $f : \Omega \to \mathbb{R}$ specifying the x_0 row of the matrix where $x_0 \in \Omega$ is a fixed basepoint (the choice of the basepoint is irrelevant—for Schurian schemes, this is due to transitivity of the G action). This is a partially symmetrized representation of $\mathbf{M}$ which we call a "spherical function".

To fully remove the symmetry of the association scheme, we represent the matrix or spherical function by its distinct entries using a function $f : \mathcal{R} \to \mathbb{R}$. Furthermore, there is a "dual", "Fourier", or "spectral" representation of a matrix in the scheme by its eigenvalues (scaled down by $\frac{1}{|\Omega|}$ by convention). This is written $\widehat{f} : \widehat{\mathcal{R}} \to \mathbb{R}$ where $\widehat{\mathcal{R}}$ are the mutual eigenspaces of the scheme (i.e. the common refinement of the eigenspaces for all matrices in the scheme, which exists by commutativity).

The first main theorem of association scheme theory is that $|\mathcal{R}| = |\widehat{\mathcal{R}}|$. This parameter is known as the *dimension* of the scheme (not to be confused with the dimension of the matrices in the scheme, which is $|\Omega|$).

Proposition 2.3. $|\mathcal{R}| = |\widehat{\mathcal{R}}|$

To deduce this result, we observe that in addition to the basis $\{\mathbf{M}_r\}_{r\in\mathcal{R}}$ there is a second natural basis for the Bose–Mesner algebra known as the *primitive idempotents*.

Definition 2.4 (Primitive idempotent, $\mathbf{E}_\alpha$). *For $\alpha \in \widehat{\mathcal{R}}$, let $\mathbf{E}_\alpha \in \mathbb{R}^{\Omega\times\Omega}$ be the projection matrix to the α eigenspace.*

Proposition 2.5. *The matrices $\mathbf{E}_\alpha$ for $\alpha \in \widehat{\mathcal{R}}$ are a basis for the Bose–Mesner algebra.*

Proof. The matrices $\mathbf{E}_\alpha$ for $\alpha \in \widehat{\mathcal{R}}$ are linearly independent and span the Bose–Mesner algebra by the commutativity assumption. It remains to prove that $\mathbf{E}_\alpha$ is in the Bose–Mesner algebra. For each pair of distinct eigenspaces $\alpha, \beta \in \widehat{\mathcal{R}}$, choose a matrix $\mathbf{A}_{\alpha,\beta}$ in the Bose–Mesner algebra which has different eigenvalues on α and β. By adding a multiple of the identity matrix (which is always in the Bose–Mesner algebra), we may assume that $\mathbf{A}_{\alpha,\beta}$ is zero on β and nonzero on α. Since the Bose–Mesner algebra is closed under matrix multiplication, the product $\prod_{\beta\in\widehat{\mathcal{R}},\beta\neq\alpha} \mathbf{A}_{\alpha,\beta}$ remains in the algebra, and since this matrix is only nonzero on α, it is a scalar multiple of $\mathbf{E}_\alpha$. We conclude that the $\mathbf{E}_\alpha$ are in the Bose–Mesner algebra and hence form a basis for it. □

Comparing the dimensions of the $\mathbf{M}_r$ basis and the $\mathbf{E}_\alpha$ basis proves Proposition 2.3. For a matrix $\mathbf{M}$ in the scheme, letting $f : \mathcal{R} \to \mathbb{R}$ be its distinct entries and $\widehat{f} : \widehat{\mathcal{R}} \to \mathbb{R}$ be its eigenvalues scaled down by $\frac{1}{|\Omega|}$, we have the correspondence,

$$\frac{1}{|\Omega|}\mathbf{M} = \frac{1}{|\Omega|}\sum_{r\in\mathcal{R}} f(r)\mathbf{M}_r = \sum_{\alpha\in\widehat{\mathcal{R}}} \widehat{f}(\alpha)\mathbf{E}_\alpha\,. \tag{10}$$

2.3 Fourier transform and P and Q functions

We have seen that functions $f : \mathcal{R} \to \mathbb{R}$ or $h : \widehat{\mathcal{R}} \to \mathbb{R}$ represent matrices in the association scheme. Define the Fourier transform (a.k.a. "MacWilliams transform") and inverse Fourier transform for the association scheme to be the mappings between these two representations.[6]

Definition 2.6 (Fourier transform and inverse Fourier transform). *For $\mathbf{M} \in \mathbb{R}^{\Omega\times\Omega}$ which is represented by $f : \mathcal{R} \to \mathbb{R}$ and dually represented by $h : \widehat{\mathcal{R}} \to \mathbb{R}$, define the Fourier transform and inverse Fourier transform by*

$$\widehat{f} = h\,, \qquad\qquad \widetilde{h} = f\,.$$

In general, the inverse Fourier transform is distinct from the Fourier transform. Note that there may not be an association scheme structure on $\widehat{R}$ and so we should be careful about which objects live on which side.

The Fourier transform and inverse Fourier transform can be interpreted as changing between the $\mathbf{M}_r$ and $\mathbf{E}_\alpha$ bases. We (definitionally) make them explicit with *P and Q functions*.

[6]Fourier analysis over a finite Abelian group G corresponds to the Schurian scheme with $\Omega = G$ acting on itself (although the scheme is not symmetric so the eigenvectors/Fourier characters are complex).

Definition 2.7 (P and Q functions)**.** *For $r \in \mathcal{R}$ and $\alpha \in \widehat{\mathcal{R}}$, define $P_r : \widehat{\mathcal{R}} \to \mathbb{R}$ and $Q_\alpha : \mathcal{R} \to \mathbb{R}$ by*

$$P_r(\alpha) = \textit{eigenvalue of } \mathbf{M}_r \textit{ on the } \alpha \textit{ eigenspace} \qquad\qquad Q_\alpha(r) = |\Omega| \cdot (r \textit{ entries of } \mathbf{E}_\alpha)\,.$$

Also define the matrices $\mathbf{P} \in \mathbb{R}^{\widehat{\mathcal{R}} \times \mathcal{R}}$ and $\mathbf{Q} \in \mathbb{R}^{\mathcal{R} \times \widehat{\mathcal{R}}}$ by $\mathbf{P}[\alpha, r] = P_r(\alpha)$ and $\mathbf{Q}[r, \alpha] = Q_\alpha(r)$.

Proposition 2.8. *For $f : \mathcal{R} \to \mathbb{R}$ and $h : \widehat{\mathcal{R}} \to \mathbb{R}$,*

$$\widehat{f}(\alpha) = \frac{1}{|\Omega|} \sum_{r \in \mathcal{R}} f(r) P_r(\alpha), \qquad\qquad \widetilde{h}(r) = \sum_{\alpha \in \widehat{\mathcal{R}}} h(\alpha) Q_\alpha(r)$$

$$h(\alpha) = \frac{1}{|\Omega|} \sum_{r \in \mathcal{R}} \widetilde{h}(r) P_r(\alpha), \qquad\qquad f(r) = \sum_{\alpha \in \widehat{\mathcal{R}}} \widehat{f}(\alpha) Q_\alpha(r)\,.$$

Viewed as vectors,

$$\widehat{f} = \frac{1}{|\Omega|} \mathbf{P} f \qquad\qquad \widetilde{h} = \mathbf{Q} h$$

Proof. The P and Q functions are the basis change coefficients so that

$$\mathbf{M}_r = \sum_{\alpha \in \widehat{\mathcal{R}}} P_r(\alpha) \mathbf{E}_\alpha \qquad\qquad \mathbf{E}_\alpha = \frac{1}{|\Omega|} \sum_{r \in \mathcal{R}} Q_\alpha(r) \mathbf{M}_r\,.$$

The equalities now follow by definition. It may help to see Eq. (10). $\square$

Corollary 2.9. $\frac{1}{|\Omega|}\mathbf{PQ} = \frac{1}{|\Omega|}\mathbf{QP} = \mathbf{Id}$.

$\mathbf{P}$ and $\mathbf{Q}$ are known respectively as the "first and second eigenmatrices" of the scheme. We can see in Proposition 2.8 that the Fourier transform of f gives its expansion in the basis of Q functions, and dually for the inverse Fourier transform in the basis of P functions. In this sense the P and Q functions play the role of the basis of Fourier characters. It is easy to get mixed up between P and Q. In a common matrix analysis set-up like in Section 1, we have a matrix represented entrywise by $f : \mathcal{R} \to \mathbb{R}$ and the goal is to compute its spectrum/Fourier transform by analyzing f in the basis of Q-functions.

There is a natural inner product structure for which the P and Q functions are orthogonal functions. The inner products are with respect to two counting measures, denoted ν (the "valencies") and μ (the "multiplicities").

Definition 2.10 (ν and μ)**.** *Define the measures ν on $\mathcal{R}$ and μ on $\widehat{\mathcal{R}}$ by*

$$\nu(r) = |\{x \in \Omega : (x_0, x) \in r\}| \qquad\qquad \mu(\alpha) = \dim(\alpha)$$

where we fix any $x_0 \in \Omega$.

In a Schurian scheme, the measure ν is independent of the choice of $x_0 \in \Omega$ due to transitivity of the action of G. The total measure of the two spaces is $\nu(\mathcal{R}) = \mu(\widehat{\mathcal{R}}) = |\Omega|$.

Equip the three spaces $\mathbf{M} \in \mathbb{R}^{\Omega\times\Omega}$, $f : \mathcal{R} \to \mathbb{R}$ and $h : \widehat{\mathcal{R}} \to \mathbb{R}$ with inner products:

$$\langle \mathbf{M}, \mathbf{N} \rangle := \operatorname{tr}(\mathbf{M}\mathbf{N}) = \sum_{x,y\in\Omega} \mathbf{M}[x,y]\mathbf{N}[x,y]$$
$$\mathop{\mathbb{E}}_{r\sim\nu} f(r)g(r) := \frac{1}{|\Omega|} \sum_{r\in\mathcal{R}} \nu(r) f(r) g(r)$$
$$\langle h, i \rangle_\mu := \sum_{\alpha\in\widehat{\mathcal{R}}} \mu(\alpha) h(\alpha) i(\alpha)$$

The first inner product on the matrix space is the Frobenius (or entrywise) inner product, which is induced by the uniform measure on $\Omega \times \Omega$. The other inner products are in fact the same inner product for the different representations of the scheme. This result generalizes the *Plancherel theorem* in Fourier analysis.

Proposition 2.11. *For matrices* $\mathbf{M}, \mathbf{N} \in \mathbb{R}^{\Omega\times\Omega}$ *with entries* $f, g : \mathcal{R} \to \mathbb{R}$*, we have* $\frac{1}{|\Omega|^2}\langle \mathbf{M}, \mathbf{N} \rangle = \mathbb{E}_{r\sim\nu} f(r)g(r) = \langle \widehat{f}, \widehat{g} \rangle_\mu$.

Proof. The first equality is:

$$\frac{1}{|\Omega|^2}\langle \mathbf{M}, \mathbf{N} \rangle = \frac{1}{|\Omega|^2} \sum_{x,y\in\Omega} \mathbf{M}[x,y]\mathbf{N}[x,y] = \frac{1}{|\Omega|} \sum_{r\in\mathcal{R}} \nu(r) f(r) g(r) = \mathop{\mathbb{E}}_{r\sim\nu} f(r) g(r)\,.$$

For the second equality: start from $\frac{1}{|\Omega|^2}\langle \mathbf{M}, \mathbf{N} \rangle = \frac{1}{|\Omega|^2} \operatorname{tr}(\mathbf{M}\mathbf{N})$. The trace is invariant under change of basis. Changing to an eigenbasis in which both $\mathbf{M}$ and $\mathbf{N}$ are diagonal, we have $\frac{1}{|\Omega|^2} \operatorname{tr}(\mathbf{M}\mathbf{N}) = \sum_{\alpha\in\widehat{\mathcal{R}}} \mu(\alpha) \widehat{f}(\alpha) \widehat{g}(\alpha) = \langle \widehat{f}, \widehat{g} \rangle_\mu$. □

We derive that the P and Q functions are orthogonal under $\langle \cdot, \cdot \rangle_\mu$ and $\mathbb{E}_{r\sim\nu}$.

Proposition 2.12.

$$\langle P_r, P_s \rangle_\mu = \langle \mathbf{M}_r, \mathbf{M}_s \rangle = \begin{cases} |\Omega|\nu(r) & r = s \\ 0 & r \neq s \end{cases} \qquad \mathop{\mathbb{E}}_{r\sim\nu} Q_\alpha(r) Q_\beta(r) = \langle \mathbf{E}_\alpha, \mathbf{E}_\beta \rangle = \begin{cases} \mu(\alpha) & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

Proof. $\langle P_r, P_s \rangle_\mu = \langle \mathbf{M}_r, \mathbf{M}_s \rangle$ and $\mathbb{E}_{r\sim\nu} Q_\alpha(r) Q_\beta(r) = \langle \mathbf{E}_\alpha, \mathbf{E}_\beta \rangle$ by the Plancherel theorem. The second equalities are direct computation. □

Corollary 2.13. $\widehat{f}(\alpha) = \frac{1}{\mu(\alpha)} \mathbb{E}_{r\sim\nu} f(r) Q_\alpha(r)$ *and* $\widetilde{h}(r) = \frac{1}{\nu(r)} \langle h, P_r \rangle_\mu$

Proof. In Proposition 2.8, take the inner product between f and Q_α or between h and P_r. Then use the orthogonality relations in Proposition 2.12. □

A "polynomial scheme" is one where the P and Q functions are univariate polynomials (with respect to P_1 and Q_1 for some ordering of $\mathcal{R}$ and $\widehat{\mathcal{R}}$). In a polynomial scheme, the P and Q functions are orthogonal polynomials for the measures μ and ν and hence additional tools can be borrowed from the theory of orthogonal polynomials. Many of the most important examples are polynomial schemes, including the Hamming and Johnson schemes.

2.4 Product structure of an association scheme

So far, we have studied the linear algebraic structure of an association scheme, identifying two natural bases $\mathbf{M}_r$ and $\mathbf{E}_\alpha$. The scheme has additional algebraic structure coming from product operations. There are two distinct products: matrix multiplication and the Hadamard product (a.k.a. entrywise product), denoted $\mathbf{M} \odot \mathbf{N}$. The Bose–Mesner algebra is closed under these products since $\mathbf{M}_i \odot \mathbf{M}_j = 1_{i=j}\mathbf{M}_i$ and $\mathbf{E}_i\mathbf{E}_j = 1_{i=j}\mathbf{E}_i$.

Definition 2.14. *Define $p_{ij}^k \in \mathbb{R}$ (the "intersection numbers") and $q_{ij}^k \in \mathbb{R}$ (the "Krein parameters") to be the values such that $\mathbf{M}_i\mathbf{M}_j = \sum_{k\in\mathcal{R}} p_{ij}^k \mathbf{M}_k$ and $\mathbf{E}_i \odot \mathbf{E}_j = \frac{1}{|\Omega|}\sum_{k\in\widehat{\mathcal{R}}} q_{ij}^k \mathbf{E}_k$.*

Proposition 2.15. *$p_{ij}^k \in \mathbb{Z}$ and $q_{ij}^k \in \mathbb{R}$. Furthermore, $0 \le p_{ij}^k \le |\Omega|$ and $0 \le q_{ij}^k \le |\Omega|$.*

Proof. The facts on p_{ij}^k follow from $p_{ij}^k = |\{y \in \Omega : (x,y) \in r_i,\ (y,z) \in r_j\}|$ in Definition 2.1.

The q_{ij}^k are the eigenvalues of the matrix $\mathbf{E}_i \odot \mathbf{E}_j$ scaled up by $|\Omega|$. We can bound them using a trick, which is that the Kronecker product $\mathbf{E}_i \otimes \mathbf{E}_j$ contains $\mathbf{E}_i \odot \mathbf{E}_j$ as a principal submatrix. The eigenvalues of $\mathbf{E}_i$ and $\mathbf{E}_j$ are either 0 or 1 and therefore all of the eigenvalues of $\mathbf{E}_i \otimes \mathbf{E}_j$ are also either 0 or 1. From this and the trick we get a bound on the quadratic form achieved by any unit vector $v \in \mathbb{R}^\Omega$:

$$0 \le \min_{\substack{w\in\mathbb{R}^{\Omega^2}\\ \|w\|_2=1}} w^\top(\mathbf{E}_i \otimes \mathbf{E}_j)w \le v^\top(\mathbf{E}_i \odot \mathbf{E}_j)v \le \max_{\substack{w\in\mathbb{R}^{\Omega^2}\\ \|w\|_2=1}} w^\top(\mathbf{E}_i \otimes \mathbf{E}_j)w \le 1\,.$$

□

Matrix multiplication and Hadamard product correspond to convolution and pointwise product on the space of functions $f : \mathcal{R} \to \mathbb{R}$, respectively. Because of this, we can compute products in either the function space or matrix space, whichever is more convenient, freely moving back and forth during the analysis. Functions in the dual space have a distinct notion of "dual convolution".

Definition 2.16 (Convolution and dual convolution)**.** *Let $\mathbf{M}, \mathbf{N} \in \mathbb{R}^{\Omega\times\Omega}$ be represented by $f, g : \mathcal{R} \to \mathbb{R}$ and dually represented by $h, i : \widehat{\mathcal{R}} \to \mathbb{R}$. Define the convolution $f * g : \mathcal{R} \to \mathbb{R}$ and the dual convolution $h \circledast i : \widehat{\mathcal{R}} \to \mathbb{R}$ by*

$$(f * g)(r) = \frac{1}{|\Omega|} \cdot (r \text{ entries of } \mathbf{MN})$$
$$(h \circledast i)(\alpha) = \text{eigenvalue of } \mathbf{M} \odot \mathbf{N} \text{ on the } \alpha \text{ eigenspace}$$

Proposition 2.17.

$$P_iP_j = \sum_{k\in\mathcal{R}} p_{ij}^k P_k \qquad\qquad P_i \circledast P_j = 1_{i=j}P_i$$
$$Q_iQ_j = \sum_{k\in\widehat{\mathcal{R}}} q_{ij}^k Q_k \qquad\qquad Q_i * Q_j = 1_{i=j}Q_i$$

Proof. In the top line, the first equation can be interpreted as computing the eigenvalues on both sides of the equation $\mathbf{M}_i\mathbf{M}_j = \sum_{k\in\mathcal{R}} p_{ij}^k \mathbf{M}_k$. The second equation can be interpreted as computing the eigenvalues of $\mathbf{M}_i \odot \mathbf{M}_j = 1_{i=j}\mathbf{M}_i$.

In the bottom line, the first equation can be interpreted as computing the entries on both sides of the equation $\mathbf{E}_i \odot \mathbf{E}_j = \frac{1}{|\Omega|} \sum_{k \in \widehat{\mathcal{R}}} q_{ij}^k \mathbf{E}_k$. The second equation can be interpreted as computing the entries of $\mathbf{E}_i \mathbf{E}_j = 1_{i=j} \mathbf{E}_i$. □

The *convolution theorem* states that the Fourier transform converts convolution in one domain to multiplication in the other domain.

Proposition 2.18. $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$ *and* $\widetilde{h \circledast i} = \widetilde{h} \cdot \widetilde{i}$.

Proof. Let $\mathbf{M}, \mathbf{N} \in \mathbb{R}^{\Omega \times \Omega}$ be the matrices represented by f and g, and dually represented by h and i. Then $f * g$ represents entrywise the matrix $\frac{1}{|\Omega|} \mathbf{MN}$. The first equality computes the eigenvalues of this matrix product.

In the second equality, both sides compute the entries of $\mathbf{M} \odot \mathbf{N}$. □

The final proposition is an explicit convolution formula.

Proposition 2.19. $(f * g)(k) = \frac{1}{|\Omega|} \sum_{i,j \in \mathcal{R}} p_{ij}^k f(i) g(j)$ *and* $(h \circledast i)(\gamma) = \sum_{\alpha,\beta \in \widehat{\mathcal{R}}} q_{\alpha\beta}^\gamma h(\alpha) i(\beta)$

Proof. Let $\mathbf{M}, \mathbf{N} \in \mathbb{R}^{\Omega \times \Omega}$ be the matrices represented by f and g, and dually represented by h and i. Then:

$$\begin{aligned}
\mathbf{MN} &= \left(\sum_{i \in \mathcal{R}} f(i) \mathbf{M}_i \right) \left(\sum_{j \in \mathcal{R}} g(j) \mathbf{M}_j \right) \\
&= \sum_{i,j \in \mathcal{R}} f(i) g(j) \sum_{k \in \mathcal{R}} p_{ij}^k \mathbf{M}_k && \text{(Definition 2.14)} \\
&= \sum_{k \in \mathcal{R}} \left(\sum_{i,j \in \mathcal{R}} p_{ij}^k f(i) g(j) \right) \mathbf{M}_k \,.
\end{aligned}$$

The formula for $f * g$ is the interpretation of this equality entrywise. Dually,

$$\begin{aligned}
\mathbf{M} \odot \mathbf{N} &= \left(\sum_{\alpha \in \widehat{\mathcal{R}}} h(\alpha) \mathbf{E}_\alpha \right) \odot \left(\sum_{\beta \in \widehat{\mathcal{R}}} i(\beta) \mathbf{E}_\beta \right) \\
&= \sum_{\alpha,\beta \in \widehat{\mathcal{R}}} h(\alpha) i(\beta) \sum_{\gamma \in \widehat{\mathcal{R}}} q_{\alpha\beta}^\gamma \mathbf{E}_\gamma && \text{(Definition 2.14)} \\
&= \sum_{\gamma \in \widehat{\mathcal{R}}} \left(\sum_{\alpha,\beta \in \widehat{\mathcal{R}}} q_{\alpha\beta}^\gamma h(\alpha) i(\beta) \right) \mathbf{E}_\gamma \,.
\end{aligned}$$

The formula for $h \circledast i$ interprets the eigenvalues of both sides. □

2.5 More about P and Q

We have been discussing the P and Q functions as if they are two distinct families of functions, but in fact they are quite nearly the same. Both are rescalings of the same bivariate function which we denote by K (for "kernel").

Definition 2.20. *Define* $K : \mathcal{R} \times \widehat{\mathcal{R}} \to \mathbb{R}$ *by* $K(r, \alpha) = \langle \mathbf{M}_r, \mathbf{E}_\alpha \rangle$*. Also define the matrix* $\mathbf{K} \in \mathbb{R}^{\mathcal{R} \times \widehat{\mathcal{R}}}$ *by* $\mathbf{K}[r, \alpha] = K(r, \alpha)$*.*

The following equation is sometimes called the "reciprocity formula" for P and Q.

Proposition 2.21. $K(r, \alpha) = \mu(\alpha) P_r(\alpha) = \nu(r) Q_\alpha(r)$

Proof. For the first equality,

$$K(r, \alpha) = \langle \mathbf{M}_r, \mathbf{E}_\alpha \rangle = P_r(\alpha) \langle \mathbf{E}_\alpha, \mathbf{E}_\alpha \rangle = \mu(\alpha) P_r(\alpha) .$$

For the second equality, both $K(r, \alpha) = \langle \mathbf{M}_r, \mathbf{E}_\alpha \rangle$ and $\nu(r) Q_\alpha(r)$ sum the elements of $\mathbf{E}_\alpha$ in relation r. □

Corollary 2.22. $\mathbf{K} = \mathbf{P}^\top \mathbf{D}_\mu = \mathbf{D}_\nu \mathbf{Q}$ *where* $\mathbf{D}_\nu$ *and* $\mathbf{D}_\mu$ *are diagonal matrices containing the values of* ν *and* μ *respectively.*

Suppose that we fix any orthogonal matrix $\mathbf{\Sigma} \in \mathbb{R}^{\Omega \times \Omega}$ whose columns are a basis of eigenvectors for the scheme. For a matrix $\mathbf{M}$ in the scheme, conjugation by $\mathbf{\Sigma}$ diagonalizes the matrix ($\widehat{\mathbf{M}} := \mathbf{\Sigma}^\top \mathbf{M} \mathbf{\Sigma}$) so we interpret conjugation by $\mathbf{\Sigma}$ as the matrix version of the Fourier transform, and conjugation by $\mathbf{\Sigma}^\top$ as the inverse Fourier transform.

One way to calculate P, Q, K for a scheme is to symmetrize $\mathbf{\Sigma}$ as follows.

Proposition 2.23.

$$\mathbf{P}[\alpha, r] = \sum_{(x,y) \in r} \mathbf{\Sigma}[x, j_0] \mathbf{\Sigma}[y, j_0] \qquad \mathbf{Q}[r, \alpha] = |\Omega| \sum_{j \in c(\alpha)} \mathbf{\Sigma}[x_0, j] \mathbf{\Sigma}[y_0, j]$$

$$\mathbf{K}[r, \alpha] = \sum_{(x,y) \in r} \sum_{j \in c(\alpha)} \mathbf{\Sigma}[x, j] \mathbf{\Sigma}[y, j]$$

where $c(\alpha) \subseteq \Omega$ *denotes the columns of* $\mathbf{\Sigma}$ *spanning the* α *eigenspace, and we fix any* $(x_0, y_0) \in r$ *and any* $j_0 \in c(\alpha)$*.*

Proof. Let $\mathbf{\Pi}_j \in \mathbb{R}^{\Omega \times \Omega}$ be the projection matrix to the jth eigenvector, which has entries $\mathbf{\Pi}_j[x, y] = \mathbf{\Sigma}[x, j] \mathbf{\Sigma}[y, j]$. Fix any $(x_0, y_0) \in r$ and $j_0 \in c(\alpha)$. Then:

$$\mathbf{K}[r, \alpha] = \langle \mathbf{M}_r, \mathbf{E}_\alpha \rangle = \sum_{j \in c(\alpha)} \langle \mathbf{M}_r, \mathbf{\Pi}_j \rangle = \sum_{(x,y) \in r} \sum_{j \in c(\alpha)} \mathbf{\Sigma}[x, j] \mathbf{\Sigma}[y, j]$$

$$P_r(\alpha) = \langle \mathbf{M}_r, \mathbf{\Pi}_{j_0} \rangle = \sum_{(x,y) \in r} \mathbf{\Sigma}[x, j_0] \mathbf{\Sigma}[y, j_0]$$

$$Q_\alpha(r) = |\Omega| \mathbf{E}_\alpha[x_0, y_0] = |\Omega| \sum_{j \in c(\alpha)} \mathbf{\Pi}_j[x_0, y_0] = |\Omega| \sum_{j \in c(\alpha)} \mathbf{\Sigma}[x_0, j] \mathbf{\Sigma}[y_0, j] .$$

□

Finally, we compute a few values of the P and Q functions. Let $0 \in \mathcal{R}$ denote the identity relation (which is always a relation by assumption). Let $0 \in \widehat{\mathcal{R}}$ denote the eigenspace containing the vector $\vec{1}$ (which is always an eigenvector—for Schurian schemes, this is due to transitivity of the G action).

Proposition 2.24. $\mu(0) = 1 \qquad \nu(0) = 1$

$$P_0(\alpha) = 1 \qquad Q_0(r) = 1 \qquad P_r(0) = \nu(r) \qquad Q_\alpha(0) = \mu(\alpha)$$

Proof. To show $\mu(0) = 1$, we claim $\mathbf{E}_0 = \frac{1}{|\Omega|}\mathbf{J}$ where $\mathbf{J} \in \mathbb{R}^{\Omega\times\Omega}$ is the all-1s matrix. Since $\mathbf{J}$ is in the scheme it can be expressed in the $\mathbf{E}_\alpha$ basis as $\mathbf{J} = \sum_{\alpha\in\widehat{\mathcal{R}}} c_\alpha \mathbf{E}_\alpha$. For any $\alpha \neq 0$,

$$\begin{aligned} c_\alpha &= \frac{\langle \mathbf{J}, \mathbf{E}_\alpha\rangle}{\langle \mathbf{E}_\alpha, \mathbf{E}_\alpha\rangle} && \text{(Orthogonality of } \mathbf{E}_\alpha \text{ under the Frobenius inner product)} \\ &= \frac{\vec{1}^\intercal \mathbf{E}_\alpha \vec{1}}{\mu(\alpha)} = 0\,. && \text{(Eigenspace } \alpha \perp \text{ eigenspace containing } \vec{1}) \end{aligned}$$

This shows that $\mathbf{E}_0$ is a scalar multiple of $\mathbf{J}$ and so this eigenspace has dimension 1.

The remaining claims are easy. $\nu(0) = 1$ is evident.

$$\begin{aligned} P_0(\alpha) &= \text{eigenvalue of } \mathbf{Id} \text{ on the } \alpha \text{ eigenspace} = 1 \\ P_r(0) &= \text{eigenvalue of } \mathbf{M}_r \text{ on the eigenspace containing } \vec{1} = \frac{1}{|\Omega|}\vec{1}^\intercal \mathbf{M}_r \vec{1} = \nu(r)\,. \end{aligned}$$

Then $Q_\alpha(0) = \mu(\alpha)$ and $Q_0(r) = 1$ by reciprocity (Proposition 2.21). □

2.6 Connection to Gelfand pairs and spherical functions

Definition 2.25 (Gelfand pair)**.** *A pair of finite groups (H, K) with $H \geq K$ is a Gelfand pair if $L^2(H/K)$ is multiplicity-free as an H-representation i.e. it decomposes into a direct sum of* ***distinct*** *irreducible representations of H.*

H/K denotes the collection of left cosets hK. The action of H on $f \in L^2(H/K)$ is a left action by $(h_1 \cdot f)(h_2K) \coloneqq f(h_1h_2K)$.

We show that finite Gelfand pairs correspond to commutative Schurian association schemes and vice versa. For example, $(S_n, S_k \times S_{n-k})$ corresponds to the Johnson scheme $J(n,k)$.

Proposition 2.26. *Let group G act transitively on a finite set Ω such that the Schurian association scheme $(\Omega, \mathcal{R})$ is commutative. Let $x_0 \in \Omega$ be arbitrary. Then (G, G_{x_0}) is a Gelfand pair where G_{x_0} is the stabilizer subgroup of x_0.*

Conversely, let (H, K) be a finite Gelfand pair. Let $\Omega = H/K$. Then the action of H on Ω yields a commutative Schurian association scheme.

Proof. Since H, K are finite we have $L^2(H/K) \cong \mathbb{R}^{H/K}$. By Schur's lemma, $\mathbb{R}^{H/K}$ is multiplicity free as an H-representation if and only if the space of H-invariant maps $\mathbf{M} : \mathbb{R}^{H/K} \to \mathbb{R}^{H/K}$ is commutative. This shows the second direction. For the first direction, in a Schurian association scheme, by the orbit-stabilizer relation Ω is in correspondence with the cosets G/G_{x_0} so the association scheme exactly consists of G-invariant maps $\mathbf{M} : \mathbb{R}^{G/G_{x_0}} \to \mathbb{R}^{G/G_{x_0}}$. □

A Gelfand pair (H, K) can be used to study the following objects: (1) K-invariant functions $f : H/K \to \mathbb{R}$ which are what we have been calling "spherical functions", (2) H-invariant matrices/linear operators $\mathbf{M} : (H/K) \times (H/K) \to \mathbb{R}$ which correspond to the Bose–Mesner algebra of an association scheme, or (3) the "Hecke algebra" of K-bi-invariant functions $f : H \to \mathbb{R}$ which satisfy $f(h) = f(k_1 h k_2)$ for all $k_1, k_2 \in K$. These three types of objects are unsymmetrized in the sense that multiple entries will be equal due to symmetry under the group action; the symmetry-reduced objects in a Gelfand pair are $f : K\backslash H/K \to \mathbb{R}$ where the domain consists of double cosets of the form KhK (these can be shown to partition H).

For example, $(O(n), O(n-1))$ are an infinite Gelfand pair. The space of spherical functions consists of $f : S^{n-1} \to \mathbb{R}$ such that $f(x)$ only depends on the angle/distance from some fixed pole $x_0 \in S^{n-1}$ (historically this motivated the term "spherical function"). The spherical functions for the Gelfand pair $(\mathbb{R}^n \rtimes O(n), O(n))$ are the radial functions $f : \mathbb{R}^n \to \mathbb{R}$.

To summarize the discussion on spherical functions, we obtain a generalized Fourier transform for $f : \Omega \to \mathbb{R}$ when Ω is not necessarily a group but has a group acting on it, in special settings where there is a "Fourier basis" (the convolution operators can be simultaneously diagonalized) which can occur even if the acting group is non-abelian.

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