The Grothendieck Constant is Strictly Larger than Davie–Reeds' Bound

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Abstract

The Grothendieck constant K_G is a fundamental quantity in functional analysis, with important connections to quantum information, combinatorial optimization, and the geometry of Banach spaces. Despite decades of study, the value of K_G is unknown. The best known lower bound on K_G was obtained independently by Davie and Reeds in the 1980s. In this paper we show that their bound is not optimal. We prove that $K_G \geq K_{DR} + 10^{-12}$, where K_{DR} denotes the Davie–Reeds lower bound.

Our argument is based on a perturbative analysis of the Davie–Reeds operator. We show that every near-extremizer for the Davie–Reeds problem has $\Omega(1)$ weight on its degree-3 Hermite coefficients, and therefore introducing a small cubic perturbation increases the integrality gap of the operator.

1 Introduction

The (real) *Grothendieck constant* is defined by

$$K_G := \sup_{n \in \mathbb{N}} \sup_{A \in \mathbb{R}^{n \times n}} \frac{\displaystyle\max_{\|x_i\|_2 = 1, \|y_j\|_2 = 1} \sum_{i,j=1}^{n} A_{ij} \langle x_i, y_j \rangle}{\displaystyle\max_{x_i \in \{\pm 1\}, y_j \in \{\pm 1\}} \sum_{i,j=1}^{n} A_{ij} x_i y_j} . \tag{1}$$

Originally studied by Grothendieck [Gro56] in the context of functional analysis, the constant K_G also plays an important role in quantum mechanics and computer science. For instance, K_G characterizes the maximal violation of Bell inequalities for 2-player XOR games via the celebrated theorem of Tsirelson [Tsi80]. In theoretical computer science, Eq. (1) is interpreted as the integrality gap of a certain semidefinite program (SDP), which has intimate connections to approximation algorithms [AN04], regularity lemmas [Sze75, FK96], and the Unique Games conjecture [RS09, KN11]. We do not attempt to list all connections of K_G here and instead we refer the reader to the surveys by Pisier [Pis12] and by Khot and Naor [KN11].

The exact value of K_G is still unknown. The best current numerical bounds are approximately $1.6769 \leq K_G \leq 1.7823$. The upper bound $K_G \leq \frac{\pi}{2\ln(1+\sqrt{2})} \approx 1.7823$ was proven by Krivine [Kri77]. While this is the best upper bound on K_G numerically speaking, in a breakthrough work, Braverman et al. [BMMN13] showed that Krivine's bound could be lowered by a positive constant $\varepsilon > 0$ (the ε coming out of their proof is so minuscule that they choose not to state an explicit numerical value).

The lower bound $K_G \geq K_{DR} \approx 1.6769$ was proven independently by Davie [Dav84] and Reeds [Ree91]. Analogous to the work of Braverman et al. relative to the Krivine bound, our contribution is to give a small but positive improvement to the Davie–Reeds bound, proving that the lower bound for K_G can be improved by a constant ε. This marks the first movement of the K_G lower bound since the 1980s.

Theorem 1.1. $K_G \geq K_{DR} + 10^{-12}$.

Concurrent work. In a concurrent work, Heilman [Hei26] has proven $K_G \geq K_{DR} + 10^{-26}$ using the same perturbation strategy that we use here, with a different analysis.

1.1 What are hard instances for the Grothendieck problem?

To prove lower bounds on K_G, we need to identify matrices $A \in \mathbb{R}^{n\times n}$ that maximize the ratio in Eq. (1). Prior literature on K_G has identified a family of interesting instances which we call *Hermite projection games*. The "matrix" in a Hermite projection game is actually a linear operator on the space of functions $\mathbb{R}^n \to \mathbb{R}$ (which can be discretized into a finite-dimensional matrix at the very end). A Hermite projection game is defined to be a linear combination of the operators $\mathbf{\Pi}_d$ where $\mathbf{\Pi}_d$ projects a function to the degree-d part of its Hermite polynomial expansion. Note that we take the limit $n \to \infty$ in a Hermite projection game, but the coefficients on the matrices $\mathbf{\Pi}_d$ are constant with respect to n.

For example, $\mathbf{\Pi}_1$ achieves value $\frac{\pi}{2}$ which is known to be the largest possible value of Eq. (1) among PSD matrices, the so-called "little Grothendieck inequality". The Davie–Reeds operator which achieves the best lower bound on K_G to date is a Hermite projection game,

$$A_{DR} = \mathbf{\Pi}_1 - \lambda^* \mathbf{I}, \qquad \lambda^* \approx 0.19748\,.$$

A remarkable result of Raghavendra and Steurer shows that Hermite projection games are complete for K_G: for every $\varepsilon > 0$, there exists a Hermite projection game such that Eq. (1) is at least $K_G - \varepsilon$, regardless of the true value of K_G [RS09].

Our improvement to the Davie–Reeds construction is also a Hermite projection game, specifically

$$A = \mathbf{\Pi}_1 - \lambda^* \mathbf{I} - \varepsilon \mathbf{\Pi}_3$$

for a small constant $\varepsilon > 0$. Let us give some intuition for why we choose this perturbation of the Davie–Reeds operator using the viewpoint of K_G through nonlocal games.

The word "game" refers to the interpretation of Eq. (1) as the gap between the optimal quantum and classical strategies for a 2-player XOR game. This type of game is parameterized by a matrix $A \in \mathbb{R}^{n\times n}$. There are two players, Alice and Bob, who separately receive an index $i_A, i_B \in [n]$ such that the probability of receiving pair (i_A, i_B) is proportional to $|A_{i_A i_B}|$.[1] Their goal is to output bits $a, b \in \{\pm 1\}$ such that $ab = \mathrm{sign}(A_{i_A i_B})$.

If no communication is allowed between Alice and Bob, their optimal strategy has bias (i.e., probability of winning minus probability of losing) exactly equal to the denominator of Eq. (1). When Alice and Bob have access to a shared quantum state, instead the bias of their optimal strategy is exactly the numerator of Eq. (1). Therefore, K_G is equal to the maximum quantum-versus-classical advantage across all possible 2-player XOR games. The famous *CHSH game* is a simple example with $n = 2$ showing $K_G > 1$ (there exists a quantum strategy which is strictly better than all possible classical strategies; this is known as a "Bell inequality").

Consider three different 2-player XOR games which are Hermite projection games:

1. Alice and Bob receive n-dimensional vectors X, Y with probability proportional to $e^{-\|X\|_2^2/2 - \|Y\|_2^2/2}|\langle X, Y\rangle|$. Their goal is to output $ab = \mathrm{sign}(\langle X, Y\rangle)$.
2. Alice and Bob both receive the same n-dimensional Gaussian vector X. Their goal is to output $ab = -1$.
3. Alice and Bob play game 1 with probability $1 - p$ or game 2 with probability p.

[1] We can assume without loss of generality that the matrix A is normalized so that $\sum_{i,j=1}^{n} |A_{ij}| = 1$, since one can easily see from Eq. (1) that K_G is scale invariant.

The operator corresponding to the first game is $\mathbf{\Pi}_1$. The operator corresponding to the second game is $-\mathbf{I}$. The operator corresponding to the third game is the (rescaled) Davie–Reeds operator $(1-p)\mathbf{\Pi}_1 - p\mathbf{I}$.

The optimal classical strategy for the first game is to output $a = \operatorname{sign}(X_1)$ and $b = \operatorname{sign}(Y_1)$. The second game is trivially winnable with probability 1. The third game, however, turns out to be strictly harder than game 1 (which is good since this leads to a larger lower bound for K_G). At the intuitive level, this is because we have confused Alice and Bob by "negating" the correct answer with probability p: when X and Y are correlated vectors, we could be in game 1, in which case the goal of Alice and Bob is to answer $ab = 1$, or we could be in game 2, in which case Alice and Bob are supposed to answer $ab = -1$ instead.

Extending this intuition, we should confuse Alice and Bob even more by introducing additional alternations. For example, a candidate harder game would be $ab = \operatorname{sign}\left(f\left(\frac{\langle X,Y\rangle}{\|X\|\|Y\|}\right)\right)$ for a function $f : [-1,1] \to \mathbb{R}$ that oscillates, since Alice and Bob would need to know the angle between X and Y very precisely in order to win this game. The work of König [Kön01] suggests a concrete instance of this type, although its analysis remains an open problem. The candidate instance is an operator on functions $\mathbb{R}^n \to \mathbb{R}$ whose "entries" are (more precisely, the kernel function of the operator is)

$$A_G(X,Y) = e^{-\|X\|_2^2/2 - \|Y\|_2^2/2} \sin\langle X, Y\rangle \,.$$

A_G is similar to a Hermite projection game but it is not quite one. If we write the Hermite polynomial expansion of $\sin\langle X,Y\rangle$

$$\sin\langle X,Y\rangle = \sum_{\alpha,\beta\in\mathbb{N}^n} c_{\alpha\beta}\, \mathrm{He}_\alpha(X)\, \mathrm{He}_\beta(Y)$$

then, in A_G, the terms with $\alpha = \beta$ are a Hermite projection game, while the terms with $\alpha \neq \beta$ are additional "off-diagonal matrices" with respect to the Hermite polynomial decomposition.

It can be shown that the Hermite coefficients $c_{\alpha\alpha}$ are zero unless $|\alpha|$ is odd and they alternate sign for $|\alpha| = 1, 3, 5, 7, \ldots$ This motivates us to consider Hermite projection games of the form $\mathbf{\Pi}_1 - c_3\mathbf{\Pi}_3 + c_5\mathbf{\Pi}_5 + \cdots$. We propose to bridge the gap from the instances $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_1 - \lambda^*\mathbf{I}$ to a candidate hard instance such as A_G by steadily adding more low-degree Hermite projectors $\mathbf{\Pi}_k$ with alternating coefficients. While the quantitative improvement in Theorem 1.1 is small, the proof of Theorem 1.1 confirms the intuition behind this approach, showing that an additional alternation $\mathbf{\Pi}_3$ indeed makes the Davie–Reeds game harder.

2 Preliminaries

We start with some preliminaries on Hermite projection games.

For a measure space (Ω, μ) and a bounded linear operator $A : L^\infty(\mu) \to L^1(\mu)$, define the quantities:

$$\mathrm{val}(A) = \sup_{f,g:\Omega\to\{\pm1\}} \int_\Omega (Af)(X) g(X)\, d\mu(X)$$

$$\mathrm{sdp}(A) = \sup_{\mathcal{H}} \sup_{\substack{f,g:\Omega\to\mathcal{H} \\ \|f(X)\|_{\mathcal{H}}=1, \|g(X)\|_{\mathcal{H}}=1}} \int_\Omega \langle (Af)(X), g(X)\rangle_{\mathcal{H}}\, d\mu(X)$$

where the supremum over $\mathcal{H}$ is over real Hilbert spaces, and we lift A to an operator on $f : \Omega \to \mathcal{H}$ by applying it separately to each coordinate of $\mathcal{H}$. More precisely, select an orthonormal basis v_i for $\mathcal{H}$, represent $f(X) = \sum_i f_i(X) v_i$, and define $Af = \sum_i (Af_i)(X) v_i$. The case where μ is the discrete counting measure on $\{1, 2, \ldots, n\}$ corresponds to finite-dimensional matrices $A \in \mathbb{R}^{n\times n}$.

The boundedness of the operator $A : L^\infty(\mu) \to L^1(\mu)$ is exactly equivalent to $\mathrm{val}(A) < \infty$. On the other hand, while $\mathrm{sdp}(A)$ could ostensibly be infinite, the Grothendieck inequality states that $\mathrm{sdp}(A)$ is in fact finite and only a constant factor larger than $\mathrm{val}(A)$.

Theorem 2.1 ([Pis12, Theorems 2.4 and 2.5])**.** *Let (Ω, μ) be a measure space and let $A : L^\infty(\mu) \to L^1(\mu)$ be a bounded linear operator. Then there is a constant K such that*

$$\mathrm{sdp}(A) \le K \, \mathrm{val}(A)$$

Furthermore, the best constant K is equal to K_G defined in Eq. (1).

Specializing to the case of Gaussian measure, let $n \in \mathbb{N}$ and let $\gamma = \gamma^{(n)}$ be the n-dimensional standard Gaussian measure. The space $L^2(\gamma)$ has the Gaussian inner product $\langle f, g\rangle_\gamma = \mathbb{E}_{X\sim\gamma} f(X)g(X)$ and the corresponding norm $\|f\|_2 = \left(\mathbb{E}_{X\sim\gamma} f(X)^2\right)^{1/2}$. For a multi-index $\alpha \in \mathbb{N}^n$, let $\mathrm{He}_\alpha(X)$ be the α Hermite polynomial (probabilists' convention) which together form an orthogonal basis for $L^2(\gamma)$. For $\alpha \in \mathbb{N}^n$ define the notation $\alpha! = \prod_{i=1}^n \alpha_i!$ and $|\alpha| = \sum_{i=1}^n \alpha_i$.

Let $\mathbf{\Pi}_k : L^2(\gamma) \to \mathrm{span}\{\mathrm{He}_\alpha : |\alpha| = k\}$ be the projection operator to the degree-k Hermite polynomials. Our candidate integrality gap instances for the Grothendieck constant are *Hermite projection games* of the form

$$A = \sum_{k=0}^{\infty} c_k \mathbf{\Pi}_k$$

for $\sup_{k\in\mathbb{N}} |c_k| < \infty$. The SDP value of a Hermite projection game has a simple formula: it is just equal to the spectral norm of A.

Proposition 2.2. *Let $A = \sum_{k=0}^\infty c_k \mathbf{\Pi}_k$ be a Hermite projection game. Then*

$$\lim_{n\to\infty} \mathrm{sdp}(A) = \sup_{k\in\mathbb{N}} |c_k| \, .$$

Proof. We first prove the upper bound. Let $L^2(\gamma; \mathcal{H})$ denote the space of functions $f : \mathbb{R}^n \to \mathcal{H}$ such that $\mathbb{E}_{X\sim\gamma}\|f(X)\|_{\mathcal{H}}^2 < \infty$. The degree-$k$ Hermite subspaces are orthogonal in this space, so

$$\|Af\|^2_{L^2(\gamma;\mathcal{H})} = \sum_{k\ge 0} c_k^2 \|\mathbf{\Pi}_k f\|^2_{L^2(\gamma;\mathcal{H})} \le \left(\sup_{k\ge0} |c_k|\right)^2 \sum_{k\ge0} \|\mathbf{\Pi}_k f\|^2_{L^2(\gamma;\mathcal{H})} = \left(\sup_{k\ge0} |c_k|\right)^2 \|f\|^2_{L^2(\gamma;\mathcal{H})}.$$

For any $f : \mathbb{R}^n \to \mathcal{H}$ such that $\|f(X)\|_{\mathcal{H}} \le 1$, we have $\|f\|_{L^2(\gamma;\mathcal{H})} \le 1$. Therefore

$$\mathop{\mathbb{E}}_{X\sim\gamma} \|Af(X)\|_{\mathcal{H}} \le \|Af\|_{L^2(\gamma;\mathcal{H})} \le \sup_{k\in\mathbb{N}} |c_k|.$$

This gives an upper bound for $\mathrm{sdp}(A)$ since it holds that

$$\mathrm{sdp}(A) = \sup_{\mathcal{H}} \sup_{\substack{f,g:\Omega\to\mathcal{H} \\ \|f(X)\|_{\mathcal{H}}\le1, \|g(X)\|_{\mathcal{H}}\le1}} \mathop{\mathbb{E}}_{X\sim\gamma} \langle (Af)(X), g(X)\rangle_{\mathcal{H}} = \sup_{\mathcal{H}} \sup_{\substack{f:\Omega\to\mathcal{H} \\ \|f(X)\|_{\mathcal{H}}\le1}} \mathop{\mathbb{E}}_{X\sim\gamma} \|(Af)(X)\|_{\mathcal{H}} \, .$$

For the lower bound, fix some constant k. For $n \ge k$, let $\mathcal{H}$ be the Hilbert space with orthonormal basis $\{e_S : S \subseteq [n], \, |S| = k\}$, and define

$$\Psi(x) := \binom{n}{k}^{-1/2} \sum_{|S|=k} \left(\prod_{i\in S} x_i\right) e_S.$$

Notice that

$$\|\Psi\|^2_{L^2(\gamma;\mathcal{H})} = \mathop{\mathbb{E}}_{X\sim\gamma} \|\Psi(X)\|^2_{\mathcal{H}} = \mathop{\mathbb{E}}_{X\sim\gamma} \binom{n}{k}^{-1} \sum_{|S|=k} \prod_{i\in S} X_i^2 = \binom{n}{k}^{-1} \sum_{|S|=k} \prod_{i\in S} \mathbb{E}\, X_i^2 = 1$$

by independence. Since $\|\Psi(X)\|_{\mathcal{H}}^2$ is the average of $\binom{n}{k}$ monomials and changing one coordinate affects only $k/n = O_k(1/n)$ fraction of them, the Efron–Stein inequality yields $\mathrm{Var}(\|\Psi(X)\|_{\mathcal{H}}^2) = O_k(1/n)$. Thus

$$\mathop{\mathbb{E}}_{X\sim\gamma}(\|\Psi(X)\|_{\mathcal{H}} - 1)^2 \le \mathop{\mathbb{E}}_{X\sim\gamma}\left(\|\Psi(X)\|_{\mathcal{H}}^2 - 1\right)^2 \to 0. \tag{2}$$

That is to say, $\|\Psi(X)\|_{\mathcal{H}}$ converges to 1 in L^2. Now define

$$f(x) := \begin{cases} \frac{\Psi(x)}{\|\Psi(x)\|_{\mathcal{H}}} & \|\Psi(x)\|_{\mathcal{H}} \ne 0, \\ 0 & \|\Psi(x)\|_{\mathcal{H}} = 0. \end{cases}$$

Then f is a feasible solution for the SDP. Moreover, $\|f - \Psi\|_{L^2(\gamma;\mathcal{H})}^2 = \mathbb{E}_{X\sim\gamma}(\|\Psi(X)\|_{\mathcal{H}} - 1)^2$ converges to 0 by Eq. (2), which implies that the Hermite weights of f converge to those of Ψ:

$$\|\mathbf{\Pi}_k f\|_{L^2(\gamma;\mathcal{H})}^2 \to \|\mathbf{\Pi}_k \Psi\|_{L^2(\gamma;\mathcal{H})}^2 = \|\Psi\|_{L^2(\gamma;\mathcal{H})}^2 = 1 \qquad \text{and} \qquad \sum_{j\ne k} \|\mathbf{\Pi}_j f\|_{L^2(\gamma;\mathcal{H})}^2 \to 0. \tag{3}$$

By Cauchy–Schwarz and the pointwise bound $\|f(x)\|_{\mathcal{H}} \le 1$, we have

$$\mathrm{sign}(c_k)\,\langle Af(X), f(X)\rangle_{\mathcal{H}} \le \|Af(X)\|_{\mathcal{H}}.$$

Taking expectations, it follows that

$$\begin{aligned} \mathop{\mathbb{E}}_{X\sim\gamma}\|Af(X)\|_{\mathcal{H}} &\ge \mathrm{sign}(c_k) \mathop{\mathbb{E}}_{X\sim\gamma}\langle Af(X), f(X)\rangle_{\mathcal{H}} \\ &= \mathrm{sign}(c_k)\sum_{j=0}^{\infty} c_j\,\|\mathbf{\Pi}_j f\|_{L^2(\gamma;\mathcal{H})}^2 \\ &\ge |c_k|\cdot\|\mathbf{\Pi}_k f\|_{L^2(\gamma;\mathcal{H})}^2 - \left(\sup_{\kappa\ge 0}|c_\kappa|\right)\sum_{j\ne k}\|\mathbf{\Pi}_j f\|_{L^2(\gamma;\mathcal{H})}^2. \end{aligned}$$

By Equation (3), the above expression converges to $|c_k|$. Since f is feasible, this is also a lower bound for the SDP value. Since this holds for all k, this completes the proof. □

Lemma 2.3. *Let A be a Hermite projection game and let $\mathrm{val}_A(f, g) = \langle Af, g\rangle_\gamma = \mathbb{E}_{X\sim\gamma}(Af)(X)g(X)$. Then $\mathrm{val}_A(f, g) = \mathrm{val}_A(f\circ T, g\circ T)$ for all orthogonal matrices $T \in O(n)$ and $f, g : \mathbb{R}^n \to \mathbb{R}$.*

Proof. We claim that this property holds more generally for operators A which are rotation-equivariant, satisfying $A(f\circ T) = (Af)\circ T$ for all $f : \mathbb{R}^n \to \mathbb{R}$ and $T \in O(n)$. To prove this,

$$\begin{aligned} \mathrm{val}_A(f\circ T, g\circ T) &= \mathop{\mathbb{E}}_{X\sim\gamma}(A(f\circ T))(X)\,g(TX) \\ &= \mathop{\mathbb{E}}_{X\sim\gamma}(Af)(TX)\,g(TX) && \text{(Rotation invariance of } A\text{)} \\ &= \mathop{\mathbb{E}}_{X'\sim\gamma}(Af)(X')\,g(X') = \mathrm{val}_A(f, g) && \text{(Rotation invariance of } \gamma\text{)} \end{aligned}$$

To use this for Hermite projection games, we note that each of the operators $\mathbf{\Pi}_k$ is rotation-equivariant. One way to prove this is using that the Ornstein–Uhlenbeck noise operator U_t decomposes as $U_t = \sum_{k=0}^{\infty} e^{-kt}\mathbf{\Pi}_k$ and the Ornstein–Uhlenbeck operator is rotation-equivariant [Bog98, Section 1.4]. Let R_T be the rotation operator $f \mapsto f\circ T$. Since R_T commutes with U_t, it preserves each eigenspace of U_t. Therefore, it preserves the space of degree-k Hermite polynomials, $R_T\mathbf{\Pi}_k = \mathbf{\Pi}_k R_T$. □

Lemma 2.4. *Let A be a Hermite projection game and $f, \widetilde{f}, g : \mathbb{R}^n \to \{\pm 1\}$. Then*

$$\left|\mathrm{val}_A(f, g) - \mathrm{val}_A(\widetilde{f}, g)\right| \le \|A\|\cdot\|f - \widetilde{f}\|_2.$$

Proof. By Cauchy–Schwarz,

$$\left| \mathop{\mathbb{E}}_{X\sim\gamma}((Af)(X) - (A\widetilde{f})(X))g(X) \right| \le \|g\|_2 \, \|A(f-\widetilde{f})\|_2 \le \|A\| \cdot \|f-\widetilde{f}\|_2$$

since $\|g\|_2 = 1$ and by definition of the operator norm. □

3 Davie–Reeds game

The Davie–Reeds game is $A_{DR} = \mathbf{\Pi}_1 - \lambda^* \mathbf{I}$ where $\lambda^* \approx 0.19748$. The constant λ^* is chosen as $\lambda^* = 2C^*\phi(C^*)$ where $C^* \approx 0.25573$ is defined as follows.

Lemma 3.1. *There is a unique solution to $4\phi(C)^2 - 4\Phi(-C) + 1 = 0$ for $C \in (0,1)$ which is $C^* \approx 0.25573$ where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and $\Phi(x) = \int_{-\infty}^{x} \phi(t)dt$.*

Proof. Let $H(C) := 4\phi(C)^2 - 4\Phi(-C) + 1$. Uniqueness of the root follows from the computations

$$H(0) = \frac{2}{\pi} - 1 < 0\,, \qquad H(1) \approx 0.59958 > 0\,, \qquad H' = 4\phi(1 - 2C\phi) > 0 \text{ for } C \in [0,1]\,.$$ □

Lemma 3.2. *Let $\widetilde{f}, \widetilde{g} : \mathbb{R}^n \to \{\pm 1\}$. Then $\widetilde{f}, \widetilde{g}$ are Davie–Reeds optimal if and only if there exists $T \in O(n)$ such that, letting $f = \widetilde{f} \circ T$ and $g = \widetilde{g} \circ T$:*

(i) Up to a set of measure zero,

$$f(X) = \begin{cases} 1 & X_1 \ge C^* \\ -g(X) & -C^* < X_1 < C^* \\ -1 & X_1 \le -C^* \end{cases} \qquad g(X) = \begin{cases} 1 & X_1 \ge C^* \\ -f(X) & -C^* < X_1 < C^* \\ -1 & X_1 \le -C^* \end{cases}$$

(ii) $\mathbf{\Pi}_1 h = 0$ where $h : \mathbb{R}^n \to \{-1,0,1\}$ is the restriction of f to the set $\{x \in \mathbb{R}^n : |x_1| \le C^*\}$.

There are many functions f, g with these two properties, see Figure 1. We will prove Lemma 3.2 in the next section. For now, we prove that all such functions have a $\mathbf{\Pi}_3$ component which is $\Omega(1)$.

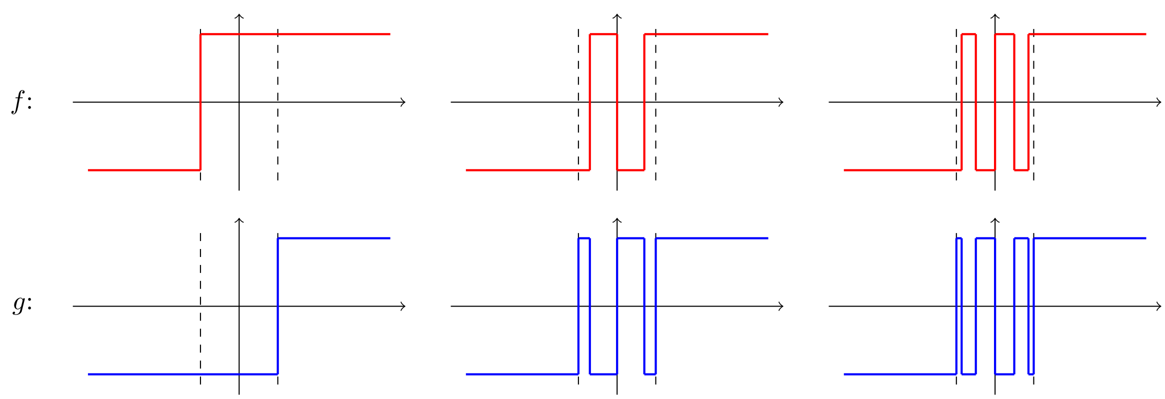

Figure 1: One-dimensional "square wave" examples of Davie–Reeds optimizers. In each example, $f = g = \operatorname{sign}(x)$ outside the strip $[-C^*, C^*]$, while $g = -f$ inside the strip, and the breakpoints are chosen so that $\mathbb{E}_{x\sim\mathcal{N}(0,1)}\, x f(x) \mathbf{1}_{|x|\le C^*} = 0$.

We say that f, g are *Davie–Reeds strip functions* if there exists $T \in O(n)$ such that $f \circ T$ and $g \circ T$ satisfy item (i) in Lemma 3.2.

Lemma 3.3. *Suppose that $f, g : \mathbb{R}^n \to \{\pm 1\}$ are Davie–Reeds strip functions. Then*

$$\mathop{\mathbb{E}}_{X\sim\gamma}(\mathbf{\Pi}_3 f)(X)(\mathbf{\Pi}_3 g)(X) \geq 0.046.$$

Proof. Applying an orthogonal change of coordinates, which does not change $\mathbb{E}_{X\sim\gamma}(\mathbf{\Pi}_3 f)(X)(\mathbf{\Pi}_3 g)(X)$ by Lemma 2.3, we may assume that f, g are strip functions in the direction X_1. Define

$$u(X) := \operatorname{sign}(X_1)\mathbf{1}_{|X_1|\geq C^*} \qquad h(X) := f(X)\mathbf{1}_{|X_1|\leq C^*}$$
$$f(X) = u(X) + h(X) \qquad g(X) = u(X) - h(X).$$

Applying $\mathbf{\Pi}_3$, we have

$$\begin{aligned}\mathop{\mathbb{E}}_{X\sim\gamma}(\mathbf{\Pi}_3 f)(X)(\mathbf{\Pi}_3 g)(X) &= \mathop{\mathbb{E}}_{X\sim\gamma}((\mathbf{\Pi}_3 u)(X) + (\mathbf{\Pi}_3 h)(X))((\mathbf{\Pi}_3 u)(X) - (\mathbf{\Pi}_3 h)(X)) \\ &= \mathop{\mathbb{E}}_{X\sim\gamma}((\mathbf{\Pi}_3 u)(X))^2 - ((\mathbf{\Pi}_3 h)(X))^2 = \|\mathbf{\Pi}_3 u\|_2^2 - \|\mathbf{\Pi}_3 h\|_2^2. \end{aligned} \tag{4}$$

We first compute $\|\mathbf{\Pi}_3 u\|_2^2$. Since u depends only on X_1, its degree-3 Hermite projection is a multiple of $\mathrm{He}_3(X_1)$ only. Thus

$$\mathbf{\Pi}_3 u = \widehat{u}(3, 0, \ldots, 0)\,\mathrm{He}_3(X_1)$$

and consequently

$$\|\mathbf{\Pi}_3 u\|_2^2 = 6\,\widehat{u}(3, 0, \ldots, 0)^2.$$

Expanding, we obtain

$$6\widehat{u}(3, 0, \ldots, 0) = \mathop{\mathbb{E}}_{X\sim\gamma} u(X)\,\mathrm{He}_3(X_1) = 2\int_{C^*}^{\infty}(x^3 - 3x)\phi(x)\,dx\,.$$

For $C^* \approx 0.25573$, this integral numerically evaluates to

$$\|\mathbf{\Pi}_3 u\|_2^2 \approx 0.0868. \tag{5}$$

Next we bound $\|\mathbf{\Pi}_3 h\|_2^2$ from above. If $\mathbf{\Pi}_3 h = 0$ there is nothing to prove, so assume otherwise. By Cauchy–Schwarz,

$$\|\mathbf{\Pi}_3 h\|_2^2 \leq \|h\|_2^2 \left\|\mathbf{1}_{|X_1|<C^*}\frac{\mathbf{\Pi}_3 h}{\|\mathbf{\Pi}_3 h\|_2}\right\|_2^2 = \|h\|_2^2\left(\mathbb{E}\,\mathbf{1}_{|X_1|<C^*}\left(\frac{\mathbf{\Pi}_3 h}{\|\mathbf{\Pi}_3 h\|_2}\right)^2\right). \tag{6}$$

We will show that

$$\mathbb{E}\,\mathbf{1}_{|X_1|<C^*}\left(\frac{\mathbf{\Pi}_3 h}{\|\mathbf{\Pi}_3 h\|_2}\right)^2 \leq \mathbb{E}\,\mathbf{1}_{|X_1|<C^*} \tag{7}$$

Towards Eq. (7), write the Hermite expansion

$$\frac{\mathbf{\Pi}_3 h}{\|\mathbf{\Pi}_3 h\|_2}(X) = \sum_{i=0}^{3}\mathrm{He}_i(X_1)\,r_i(X_2, \ldots, X_n),$$

where each r_i lies in the degree-$(3-i)$ Hermite subspace in the remaining variables. Since $\mathbf{1}_{|X_1|<C^*}$ depends only on X_1, orthogonality in $(X_2, \ldots, X_n)$ gives

$$\mathbb{E}\,\mathbf{1}_{|X_1|<C^*}\left(\frac{\mathbf{\Pi}_3 h}{\|\mathbf{\Pi}_3 h\|_2}\right)^2 = \sum_{i=0}^{3}\mathbb{E}\left(\mathbf{1}_{|X_1|<C^*}\,\mathrm{He}_i(X_1)^2\right)\|r_i\|_2^2 \leq \max_{0\leq i\leq 3}\frac{\mathbb{E}\,\mathbf{1}_{|X_1|<C^*}\,\mathrm{He}_i(X_1)^2}{i!}$$

where the inequality holds because

$$1 = \left\| \frac{\mathbf{\Pi}_3 h}{\|\mathbf{\Pi}_3 h\|_2} \right\|_2^2 = \sum_{i=0}^{3} i!\, \|r_i\|_2^2.$$

Notice that on $|x| \le C^* < 1$,

$$\frac{\mathrm{He}_0(x)^2}{0!} = 1, \qquad \frac{\mathrm{He}_1(x)^2}{1!} = x^2 \le 1, \qquad \frac{\mathrm{He}_2(x)^2}{2!} = \frac{(x^2-1)^2}{2} \le 1, \qquad \frac{\mathrm{He}_3(x)^2}{3!} = \frac{(x^3-3x)^2}{6} < 1.$$

Therefore

$$\frac{\mathbb{E}\, \mathbf{1}_{|X_1|<C^*}\, \mathrm{He}_i(X_1)^2}{i!} \le \mathbb{E}\, \mathbf{1}_{|X_1|<C^*} \qquad (0 \le i \le 3).$$

This proves Eq. (7). On the other hand, since $h = \pm 1$ on the strip and $h = 0$ otherwise,

$$\|h\|_2^2 = \mathbb{E}\, \mathbf{1}_{|X_1|<C^*} = 2\Phi(C^*) - 1 \approx 0.20184$$

Plugging this into Eq. (6) shows that,

$$\|\mathbf{\Pi}_3 h\|_2^2 \lessapprox (0.20184)^2 \approx 0.0408. \tag{8}$$

Combining Eqs. (5), (8) and (4),

$$\mathbb{E}(\mathbf{\Pi}_3 f)(X)(\mathbf{\Pi}_3 g)(X) \ge 0.0868 - 0.0408 \ge 0.046$$

This proves the lemma. □

4 Perturbing the Davie–Reeds game

For $\varepsilon, \lambda \in \mathbb{R}$ define the *perturbed Davie–Reeds game* by

$$A_{\varepsilon,\lambda} = \mathbf{\Pi}_1 - \lambda \mathbf{I} - \varepsilon \mathbf{\Pi}_3\,.$$

The Davie–Reeds game itself uses $\varepsilon = 0$ and $\lambda = \lambda^*$. We fix $\lambda = \lambda^*$ throughout and let $A_\varepsilon = A_{\varepsilon,\lambda^*}$.

Theorem 4.1 (Improvement over Davie–Reeds)**.** $\mathrm{val}(A_\varepsilon) \le \mathrm{val}(A_{DR}) - \varepsilon(0.046 - 12(2\varepsilon)^{1/4})$.

The main lemma for the proof is a stability estimate for the optimizers of the Davie–Reeds game. Let $\mathrm{val}_A(f,g) = \mathbb{E}_{X\sim\gamma}(Af)(X)g(X)$.

Lemma 4.2. *Assume $f, g : \mathbb{R}^n \to \{\pm 1\}$ have $\mathrm{val}_{A_{DR}}(f,g) \ge \mathrm{val}(A_{DR}) - \eta$. Then there exist Davie–Reeds strip functions $f_{DR}, g_{DR} : \mathbb{R}^n \to \{\pm 1\}$ such that $\|f - f_{DR}\|_2 \le 6\eta^{1/4}$ and $\|g - g_{DR}\|_2 \le 6\eta^{1/4}$.*

We can quickly deduce the theorem from the lemma.

Proof of Theorem 4.1 from Lemma 4.2. Let $f, g : \mathbb{R}^n \to \{\pm 1\}$. The value achieved by f and g is

$$\begin{aligned} \mathrm{val}_{A_\varepsilon}(f,g) &= \mathop{\mathbb{E}}_{X\sim\gamma}(\mathbf{\Pi}_1 f)(X)(\mathbf{\Pi}_1 g)(X) - \lambda^* \mathop{\mathbb{E}}_{X\sim\gamma} f(X)g(X) - \varepsilon \mathop{\mathbb{E}}_{X\sim\gamma}(\mathbf{\Pi}_3 f)(X)(\mathbf{\Pi}_3 g)(X) \\ &= \mathrm{val}_{A_{DR}}(f,g) - \varepsilon \mathop{\mathbb{E}}_{X\sim\gamma}(\mathbf{\Pi}_3 f)(X)(\mathbf{\Pi}_3 g)(X). \end{aligned} \tag{9}$$

We split into two cases. In the first case $\mathrm{val}_{A_{DR}}(f,g) \ge \mathrm{val}(A_{DR}) - 2\varepsilon$. By Lemma 4.2, there exist Davie–Reeds strip functions f_{DR}, g_{DR} such that

$$\|f - f_{DR}\|_2 \le 6(2\varepsilon)^{1/4}, \qquad \|g - g_{DR}\|_2 \le 6(2\varepsilon)^{1/4}.$$

Using Lemma 2.4 with $A = \mathbf{\Pi}_3$ twice,

$$|\mathbb{E}(\mathbf{\Pi}_3 f)(X)(\mathbf{\Pi}_3 g)(X) - \mathbb{E}(\mathbf{\Pi}_3 f_{DR})(X)(\mathbf{\Pi}_3 g_{DR})(X)| \leq \|f - f_{DR}\|_2 + \|g - g_{DR}\|_2 \leq 12(2\varepsilon)^{1/4}.$$

Hence by Lemma 3.3,

$$\mathbb{E}(\mathbf{\Pi}_3 f)(X)(\mathbf{\Pi}_3 g)(X) \geq 0.046 - 12(2\varepsilon)^{1/4}.$$

Plugging this into Eq. (9),

$$\mathrm{val}_{A_\varepsilon}(f,g) \leq \mathrm{val}(A_{DR}) - \varepsilon\left(0.046 - 12(2\varepsilon)^{1/4}\right).$$

In the second case $\mathrm{val}_{A_{DR}}(f,g) < \mathrm{val}(A_{DR}) - 2\varepsilon$. By Cauchy–Schwarz and since $\mathbf{\Pi}_3$ is an orthogonal projection,

$$\left|\mathop{\mathbb{E}}_{X\sim\gamma}(\mathbf{\Pi}_3 f)(X)(\mathbf{\Pi}_3 g)(X)\right| \leq \|\mathbf{\Pi}_3 f\|_2\|\mathbf{\Pi}_3 g\|_2 \leq \|f\|_2\|g\|_2 = 1.$$

Therefore

$$\mathrm{val}_{A_\varepsilon}(f,g) \leq \mathrm{val}(A_{DR}) - 2\varepsilon + \varepsilon = \mathrm{val}(A_{DR}) - \varepsilon.$$

Combining the two cases yields the desired theorem. ☐

Proof of Theorem 1.1. Combine Proposition 2.2 with Theorem 4.1 to get

$$K_G \geq \frac{\mathrm{sdp}(A_\varepsilon)}{\mathrm{val}(A_\varepsilon)} \geq \frac{1-\lambda^*}{\mathrm{val}(A_{DR}) - \varepsilon(0.046 - 12(2\varepsilon)^{1/4})} \geq \frac{1-\lambda^*}{\mathrm{val}(A_{DR})} + \frac{1-\lambda^*}{\mathrm{val}(A_{DR})^2}\cdot\varepsilon(0.046 - 12(2\varepsilon)^{1/4}).$$

The last inequality is using the convexity of $f(x) = \frac{1-\lambda^*}{\mathrm{val}(A_{DR})-x}$ at $x = 0$. Selecting $\varepsilon = 4\cdot 10^{-11}$ is enough to make $0.046 - 12(2\varepsilon)^{1/4} \geq 0.01$. Plug in $\lambda^* \approx 0.19748$ and $\mathrm{val}(A_{DR}) \approx 0.4786$ to obtain the lower bound $K_G \geq K_{DR} + 10^{-12}$. ☐

4.1 Stability analysis

Now we prove Lemma 4.2. First, we repeat the analysis of Davie and Reeds of their game.

Lemma 4.3 ([Dav84, Ree91])**.** $\mathrm{val}(A_{DR}) \leq 4\phi(C^*)^2 - \lambda^*(4\Phi(-C^*) - 1)$.

Proof. Recall $A_{DR} = \mathbf{\Pi}_1 - \lambda^*\mathbf{I}$. We use an arbitrary $\lambda \in (0, 0.3)$ in the analysis, explaining how to pick the optimal $\lambda = \lambda^*$ at the end.

Let $f, g : \mathbb{R}^n \to \{\pm 1\}$. First, rotate f and g so that $\mathbf{\Pi}_1 f = \sigma X_1$ for some $\sigma \in \mathbb{R}$. This does not change the value of f, g by Lemma 2.3. Let $\mu = \mathbb{E}_{X\sim\gamma} X_1 g(X)$. By the Cauchy–Schwarz inequality,

$$\begin{aligned}\mathrm{val}_{A_{DR}}(f,g) = \sigma\mu - \lambda\mathop{\mathbb{E}}_{X\sim\gamma} f(X)g(X) &\leq \left(\frac{\sigma+\mu}{2}\right)^2 - \lambda\mathop{\mathbb{E}}_{X\sim\gamma} f(X)g(X). \qquad (10)\\ &= \left(\mathop{\mathbb{E}}_{X\sim\gamma} X_1\left(\frac{f(X)+g(X)}{2}\right)\right)^2 - \lambda\mathop{\mathbb{E}}_{X\sim\gamma} f(X)g(X)\end{aligned}$$

Let $S = \{x \in \mathbb{R}^n : f(x) = g(x)\}$ and let $\gamma(S)$ denote its Gaussian measure. Then:

$$\left(\mathop{\mathbb{E}}_{X\sim\gamma} X_1\left(\frac{f(X)+g(X)}{2}\right)\right)^2 \leq \left(\mathop{\mathbb{E}}_{X\sim\gamma}|X_1|\cdot\mathbf{1}_S(X)\right)^2, \qquad \mathop{\mathbb{E}}_{X\sim\gamma} f(X)g(X) = 2\gamma(S) - 1. \qquad (11)$$

Putting these together in Eq. (10),

$$\mathrm{val}_{A_{DR}}(f,g) \le \left(\mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_S(X)\right)^2 - \lambda \cdot (2\gamma(S) - 1)\,. \tag{12}$$

For fixed $\gamma(S)$, the right-hand side is maximized by choosing S to be of the form $S = \{x \in \mathbb{R}^n : |x_1| \ge C\}$. This is because, when the measure of S is fixed, we may as well allocate the mass of S to make $|X_1|$ as large as possible. This yields the upper bound,

$$\begin{aligned} \mathrm{val}_{A_{DR}}(f,g) &\le \sup_{C\ge 0} \left(\mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_{|X_1|\ge C}\right)^2 - \lambda \cdot (4\Phi(-C) - 1) \\ &= \sup_{C\ge 0} 4\phi(C)^2 - \lambda \cdot (4\Phi(-C) - 1) \end{aligned} \tag{13}$$

where ϕ and Φ are the Gaussian PDF and CDF. Single-variable calculus can now be used to solve this optimization problem.

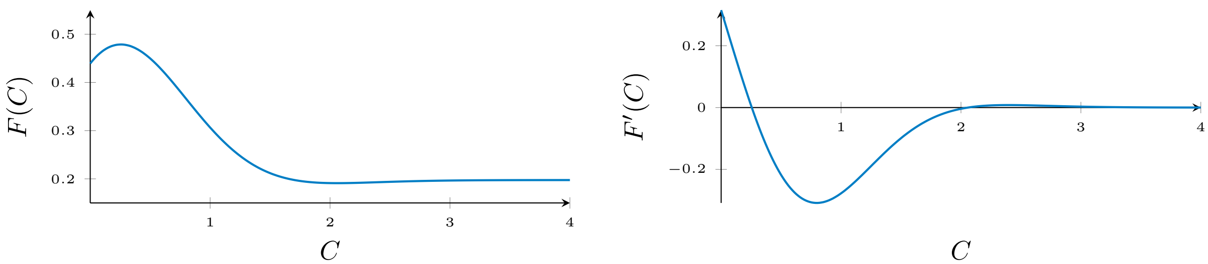

Figure 2: Plots of $F(C) := 4\phi(C)^2 - \lambda^*(4\Phi(-C) - 1)$ along with its first derivative, for $\lambda^* \approx 0.19748$. The only two roots of F' are $C_- \approx 0.25573$ and $C_+ \approx 2.0582$.

Let $F(C) := 4\phi(C)^2 - \lambda \cdot (4\Phi(-C) - 1)$. Using the properties $\Phi' = \phi$ and $\phi' = -C\phi$, the first two derivatives of F with respect to C are:

$$\begin{aligned} F' &= 8\phi\phi' + 4\lambda\phi = 4\phi\big(\lambda - 2C\phi\big) \\ F'' &= -4\lambda C\phi - 16C\phi\phi' - 8\phi^2 = -4\lambda C\phi + 8\phi^2(2C^2 - 1). \end{aligned} \tag{14}$$

Hence $F'(C) = 0$ if and only if $\lambda = 2C\phi(C)$. By taking derivatives, we see that the function $2C\phi(C)$ is strictly increasing on $(0, 1)$ and decreasing on $(1, \infty)$, with maximum value at $C = 1$. Therefore, for all $0 < \lambda < 2\phi(1) \approx 0.4839$, there are two distinct solutions to the equation $\lambda = 2C\phi(C)$ which we denote $C_- \in (0, 1)$ and $C_+ \in (1, \infty)$. The second derivative evaluated at these critical points satisfies

$$F'' = -4(2C\phi)C\phi + 8\phi^2(2C^2 - 1) = 8\phi^2(C^2 - 1)\,.$$

Since C_-, C_+ lie on opposite sides of 1, $F''(C_-) < 0$ and C_- is a local maximum, while $F''(C_+) > 0$ and C_+ is a local minimum. Therefore, the maximum value of F is either $F(C_-)$ or $\lim_{C\to\infty} F(C) = \lambda$. We have

$$F(C_-) \underset{(F'(0)>0)}{>} F(0) = \frac{2}{\pi} - \lambda \underset{(\lambda<\frac{1}{\pi}\approx 0.318)}{>} \lambda\,.$$

So $F(C_-)$ is the maximum. We obtain the upper bound

$$\mathrm{val}_{A_{DR}}(f,g) \le F(C_-) = 4\phi(C_-)^2 - \lambda(4\Phi(-C_-) - 1)\,. \tag{15}$$

This proves the desired upper bound on the value of the Davie–Reeds game. To choose $\lambda = \lambda^*$ in their game, we first choose $C^* = C_- \in (0,1)$ then set $\lambda^* = 2C^*\phi(C^*)$ correspondingly. C^* is chosen to maximize the integrality gap ratio. The reciprocal of the integrality gap ratio is:

$$\frac{\mathrm{val}(A_\lambda)}{SDP(A_\lambda)} \underset{\text{(Proposition 2.2, Eq. (15))}}{\leq} \frac{4\phi(C)^2 - \lambda(4\Phi(-C)-1)}{1-\lambda} = \frac{4\phi(C)^2 - 2C\phi(C)(4\Phi(-C)-1)}{1-2C\phi(C)} \quad (=: R(C))$$

The derivative with respect to C of this expression is

$$R'(C) = \frac{2(1-C^2)\phi(C)}{(1-2C\phi(C))^2}\left(4\phi(C)^2 - 4\Phi(-C)+1\right)$$

This motivates us to choose C to be a critical point, which satisfies $4\phi(C)^2 - 4\Phi(-C) + 1 = 0$. □

The optimizers of the Davie–Reeds game are characterized by checking when all of the inequalities are tight in the above proof. We prove this next.

Proof of Lemma 3.2. Let $f, g : \mathbb{R}^n \to \{\pm 1\}$ be Davie–Reeds optimizers. Rotate f, g so that $\mathbf{\Pi}_1 f = \sigma X_1$. Eq. (10) is tight if and only if

$$\underset{X\sim\gamma}{\mathbb{E}} X_1 f(X) = \underset{X\sim\gamma}{\mathbb{E}} X_1 g(X) = \sigma\,. \tag{16}$$

We will use this momentarily.

Let $S = \{x \in \mathbb{R}^n : f(x) = g(x)\}$. Eq. (11) is tight if and only if $\mathrm{sign}(X_1) = f(X)$ for all $X \in S$, or $\mathrm{sign}(X_1) = -f(X)$ for all $X \in S$ (up to a set of measure zero). Without of loss of generality, the former is the case, otherwise we apply $T \in O(n)$ which reflects X_1.

The argument from Eq. (12) to Eq. (13) is tight if and only if $S = \{x \in \mathbb{R}^n : |x_1| \geq C\}$ for some $C \in \mathbb{R}$. The calculus argument is tight if and only if $C = C^*$. We deduce that f, g satisfy item (i) of Lemma 3.2,

$$f(X) = \begin{cases} 1 & X_1 \geq C^* \\ -g(X) & -C^* < X_1 < C^* \\ -1 & X_1 \leq -C^* \end{cases} \qquad g(X) = \begin{cases} 1 & X_1 \geq C^* \\ -f(X) & -C^* < X_1 < C^* \\ -1 & X_1 \leq -C^* \end{cases} \tag{17}$$

Towards item (ii), write

$$\begin{aligned} & u(X) := \mathrm{sign}(X_1)\mathbf{1}_{|X_1|\geq C^*} \qquad h(X) := f(X)\mathbf{1}_{|X_1|\leq C^*} \\ & f(X) = u(X) + h(X) \qquad g(X) = u(X) - h(X) \\ \implies & \mathbf{\Pi}_1 f = \mathbf{\Pi}_1 u + \mathbf{\Pi}_1 h \qquad \mathbf{\Pi}_1 g = \mathbf{\Pi}_1 u - \mathbf{\Pi}_1 h\,. \end{aligned} \tag{18}$$

We have $\mathbf{\Pi}_1 f = \sigma X_1$ due to the initial rotation, and $\mathbf{\Pi}_1 u$ is also a multiple of X_1 since u only depends on X_1. Eq. (18) implies that $\mathbf{\Pi}_1 g, \mathbf{\Pi}_1 h$ are also multiples of X_1. On the other hand, Eq. (16) then implies

$$\mathbf{\Pi}_1 f = \mathbf{\Pi}_1 g\,. \tag{19}$$

Combining Eqs. (19) and (18) shows $\mathbf{\Pi}_1 h = 0$ which proves item (ii).

Finally, we observe that functions satisfying these properties indeed exist, for example in Fig. 1. □

To turn this into a stability argument, we start by proving two quantitative stability lemmas.

Lemma 4.4. *Let $S \subseteq \mathbb{R}^n$ and let $S^* = \{x \in \mathbb{R}^n : |x_1| \geq C\}$ where $C \geq 0$ is chosen so that $\gamma(S) = \gamma(S^*) = 2\Phi(-C)$. If $\mathbb{E}_{X\sim\gamma}|X_1| \cdot \mathbf{1}_S(X) \geq 2\phi(C) - \varepsilon$, then $\gamma(S \triangle S^*) \leq 4\sqrt{\varepsilon}$.*

Proof. The assumption can be rewritten as

$$\mathop{\mathbb{E}}_{X\sim\gamma}|X_1|\cdot\mathbf{1}_{S^*}(X)-\mathop{\mathbb{E}}_{X\sim\gamma}|X_1|\cdot\mathbf{1}_{S}(X)=\mathop{\mathbb{E}}_{X\sim\gamma}|X_1|\cdot\mathbf{1}_{S^*\setminus S}(X)-\mathop{\mathbb{E}}_{X\sim\gamma}|X_1|\cdot\mathbf{1}_{S\setminus S^*}(X)\le\varepsilon\,.$$

Since S and S^* have the same Gaussian measure, so do $S\setminus S^*$ and $S^*\setminus S$. Adding $\mathbb{E}_{X\sim\gamma}C\cdot\mathbf{1}_{S\setminus S^*}(X)-\mathbb{E}_{X\sim\gamma}C\cdot\mathbf{1}_{S^*\setminus S}(X)=0$ to the above equation, we get

$$\mathop{\mathbb{E}}_{X\sim\gamma}(|X_1|-C)\cdot\mathbf{1}_{S^*\setminus S}(X)+\mathop{\mathbb{E}}_{X\sim\gamma}(C-|X_1|)\cdot\mathbf{1}_{S\setminus S^*}(X)\le\varepsilon\,.$$

Both terms are now non-negative, so each one is at most ε. We conclude that

$$\mathop{\mathbb{E}}_{X\sim\gamma}|C-|X_1||\cdot\mathbf{1}_{S\triangle S^*}(X)\le 2\varepsilon\,.$$

Now for a parameter $t\ge 0$ we bound

$$\gamma(S\triangle S^*)\le\Pr(|C-|X_1||\le t)+\frac{1}{t}\mathop{\mathbb{E}}_{X\sim\gamma}|C-|X_1||\cdot\mathbf{1}_{S\triangle S^*}(X)\le 2\sqrt{\frac{2}{\pi}}t+\frac{2\varepsilon}{t}\,.$$

Choosing $t=\sqrt{\varepsilon}$ we get $\gamma(S\triangle S^*)\le(2\sqrt{\frac{2}{\pi}}+2)\sqrt{\varepsilon}\le 4\sqrt{\varepsilon}$. □

Lemma 4.5. *Let* $F(C)=4\phi(C)^2-\lambda^*(4\Phi(-C)-1)$ *and* $0\le\varepsilon<0.01$. *If* $F(C)\ge F(C^*)-\varepsilon$ *then* $|C-C^*|\le 3\sqrt{\varepsilon}$.

Proof. Using the derivative sign calculations in the proof of Lemma 4.3 (see Section 4.1), the superlevel sets of F have the form

$$\{C\ge 0: F(C)\ge F(C^*)-\varepsilon\}=[C_1,C_2]\cup[C_3,\infty)$$

where $C^*=C_-\in[C_1,C_2]$ and $C_+\le C_3$. The maximum value of F in the interval $[C_3,\infty)$ is at most $\lim_{C\to\infty}F(C)=\lambda^*\approx 0.19748$ so the latter interval does not appear since $F(C^*)\approx 0.4786$. We bound $0\le C_1\le C_2\le 1/2$ using $F(0)\approx 0.4391$ and $F(1/2)\approx 0.4496$.

On the interval $C\in[0,1/2]$ we bound the second derivative by, using Eq. (14),

$$F''(C)\le 8\phi(C)^2(2C^2-1)\le -4\phi(1/2)^2=-\frac{2}{\pi}e^{-1/4}<-0.49\,. \tag{20}$$

Using Taylor's theorem with remainder, for every $C\in[C_1,C_2]$ there exists $\xi\in[C_1,C_2]$ such that

$$F(C)=F(C^*)+\frac{1}{2}F''(\xi)(C-C^*)^2\,.$$

By Eq. (20), we get $F(C^*)-F(C)\ge 0.245(C-C^*)^2$. Assuming the left side is at most ε, then

$$|C-C^*|^2\le\frac{\varepsilon}{0.245}\le 9\varepsilon\implies|C-C^*|\le 3\sqrt{\varepsilon}\,.$$ □

Proof of Lemma 4.2. Assume that $f,g:\mathbb{R}^n\to\{\pm 1\}$ satisfy $\mathrm{val}_{A_{DR}}(f,g)\ge\mathrm{val}(A_{DR})-\varepsilon$. Following the proof of Lemma 4.3, we convert f,g into a nearby pair of Davie–Reeds strip functions f_{DR},g_{DR}.

First, apply a rotation so that $\mathbf{\Pi}_1 f = \sigma X_1$. Let $S = \{x \in \mathbb{R}^n : f(x) = g(x)\}$. Partition $S = S_+ \cup S_-$ by the value of $\operatorname{sign}(X_1)\left(\frac{f(X)+g(X)}{2}\right)$. Then,

$$\begin{aligned}
&\left(\mathop{\mathbb{E}}_{X\sim\gamma} X_1 \left(\frac{f(X)+g(X)}{2}\right)\right)^2 \\
&= \left(\mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_{S_+}(X) - \mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_{S_-}(X)\right)^2 \\
&= \left(\mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_{S_+}(X) + \mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_{S_-}(X)\right)^2 - 4\left(\mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_{S_+}(X)\right)\left(\mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_{S_-}(X)\right) \\
&= \left(\mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_{S}(X)\right)^2 - 4\left(\mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_{S_+}(X)\right)\left(\mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_{S_-}(X)\right).
\end{aligned}$$

Since Eq. (11) has error at most ε, we deduce that

$$4\left(\mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_{S_+}(X)\right)\left(\mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_{S_-}(X)\right) \le \varepsilon\,. \tag{21}$$

Therefore, one of the inequalities holds:

$$\begin{cases} \mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_{S_+}(X) \le \dfrac{\sqrt{\varepsilon}}{2} \\ \mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_{S_-}(X) \le \dfrac{\sqrt{\varepsilon}}{2} \end{cases} \tag{22}$$

Without loss of generality, the second inequality holds (else apply $T \in O(n)$ which reflects X_1).

Let $S^* = \{x \in \mathbb{R}^n : |x_1| \ge C^*\}$. Then since $|X_1| \ge C^*$ on S^*, we have

$$\begin{aligned}
C^*\gamma(S_- \cap S^*) &\le \mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_{S_-\cap S^*}(X) \le \mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_{S_-}(X) \underset{\text{Eq. (22)}}{\le} \frac{\sqrt{\varepsilon}}{2} \\
\implies \gamma(S_- \cap S^*) &\le \frac{\sqrt{\varepsilon}}{2C^*}
\end{aligned} \tag{23}$$

Next, let $C \ge 0$ be such that the set $S_C = \{x \in \mathbb{R}^n : |x_1| \ge C\}$ has Gaussian measure $\gamma(S_C) = \gamma(S) = 2\Phi(-C)$. The argument from Eqs. (12) and (13) established that

$$\left(\mathop{\mathbb{E}}_{X\sim\gamma} |X_1| \cdot \mathbf{1}_{S}(X)\right)^2 - \lambda \cdot (2\gamma(S) - 1) \le 4\phi(C)^2 - \lambda \cdot (4\Phi(-C) - 1) \le \sup_{C^* \ge 0} 4\phi(C^*)^2 - \lambda \cdot (4\Phi(-C^*) - 1)\,.$$

Furthermore, the loss in both inequalities is at most ε. Using Lemma 4.4 on the left inequality,

$$\gamma(S \triangle S_C) \le 4\sqrt{\varepsilon}\,. \tag{24}$$

Using Lemma 4.5 on the right inequality, we have $|C - C^*| \le 3\sqrt{\varepsilon}$. Then using $\Phi' = \phi$ we deduce

$$\gamma(S_C \triangle S^*) = 2|\Phi(-C) - \Phi(-C^*)| \le 2\left(\max_{t\in\mathbb{R}} \phi(t)\right)|C - C^*| \le 3\sqrt{\varepsilon}\,. \tag{25}$$

Combine Eq. (24) and Eq. (25) to get

$$\gamma(S \triangle S^*) \le 7\sqrt{\varepsilon} \tag{26}$$

We now define Davie–Reeds strip functions $f_{DR}, g_{DR} : \mathbb{R}^n \to \{\pm 1\}$ by

$$f_{DR}(X) = \begin{cases} \mathrm{sign}(X_1) & X \in S^* \\ -f(X) & X \in S \setminus S^* \\ f(X) & \text{otherwise} \end{cases} \qquad g_{DR}(X) = \begin{cases} \mathrm{sign}(X_1) & X \in S^* \\ g(X) & \text{otherwise} \end{cases}.$$

By construction, f_{DR}, g_{DR} are Davie–Reeds strip functions. It can be seen that the modification set of both f and g is contained within $(S \triangle S^*) \cup (S_- \cap S^*)$. Using Eqs. (23) and (26), we conclude that

$$\|f - f_{DR}\|_2^2 = 4\gamma(\{X : f(X) \neq f_{DR}(X)\}) \leq 4\gamma(S \triangle S^*) + 4\gamma(S_- \cap S^*) \leq 4\left(7 + \frac{1}{2C^*}\right)\sqrt{\varepsilon},$$

$$\|g - g_{DR}\|_2^2 = 4\gamma(\{X : g(X) \neq g_{DR}(X)\}) \leq 4\gamma(S \triangle S^*) + 4\gamma(S_- \cap S^*) \leq 4\left(7 + \frac{1}{2C^*}\right)\sqrt{\varepsilon}.$$

Since $C^* \approx 0.25573$, the right-hand side is at most $36\sqrt{\varepsilon}$, and therefore

$$\|f - f_{DR}\|_2 \leq 6\varepsilon^{1/4} \qquad \text{and} \qquad \|g - g_{DR}\|_2 \leq 6\varepsilon^{1/4}. \qquad \square$$

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