Spherical Discrepancy Minimization and Algorithmic Lower Bounds for Covering the Sphere

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Outline

Introduction

Our Algorithm

Applications to Covering Problems

Open Problems

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The Spherical Discrepancy Problem

Spherical Discrepancy Problem

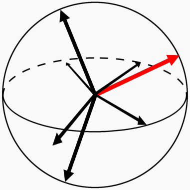
Given unit vectors $u_1, \ldots, u_m \in S^n$, find $x \in S^n$ which minimizes

$$\max_i \langle u_i, x \rangle .$$

Think of $m = \text{poly}(n)$ or $m = 2^{\sqrt{n}}$.
If $m < n$, OPT ≤ 0 by Gaussian elimination.

Solving exactly (or even approximately) is NP-hard.

We look for a good output **independent of the true optimum**.



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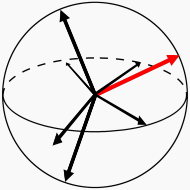


Solving exactly (or even approximately) is NP-hard.

We look for a good output **independent of the true optimum**.

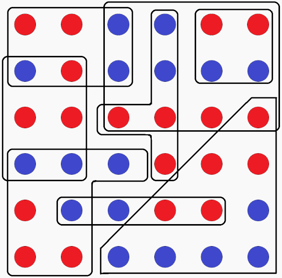
Baseline algorithm: random choice of x.

$x \in_R S^n$ implies $\langle u_i, x \rangle \approx 1/\sqrt{n}$. **We will normalize to** $\|x\|_2 = \sqrt{n}$.

Exponential-time, practical algorithm given by Petković et al

Motivation: The Boolean Discrepancy Problem

Given a collection of subsets of $[n]$, color $[n]$ red or blue so that each set is as balanced as possible (minimize the "discrepancy" between red and blue).



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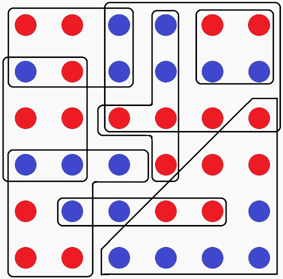

Reframing: let v_i be the 0/1 indicator of the i-th set. Find $x \in \{\pm 1\}^n$ which minimizes

$$\max_i |\langle v_i, x \rangle|$$

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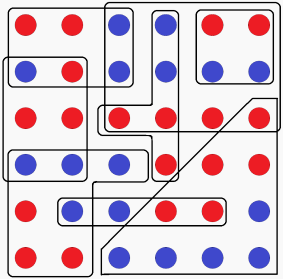

Boolean Discrepancy Problem

Given unit vectors $u_1, \ldots, u_m \in S^n$, find $x \in \{\pm 1\}^n$ which minimizes

$$\max_i |\langle u_i, x \rangle|$$

Spherical Discrepancy Problem

Given unit vectors $u_1, \ldots, u_m \in S^n$, find x s.t. $\|x\|_2 = \sqrt{n}$ which minimizes

$$\max_i \langle u_i, x \rangle .$$

The absolute value sign is essentially ignorable.

Algorithms for Boolean and Spherical Discrepancy

Given unit vectors $u_1, \ldots, u_m \in S^n$, find x which minimizes

$$\max_{i \in [m]} \langle u_i, x \rangle$$

$(m = O(n))$

	$x \in \{\pm 1\}^n$	$\|x\|_2 = \sqrt{n}$
Random x	$O(\sqrt{\log n})$	$O\left(\sqrt{\log n}\right)$
Partial coloring [S'85, LM'12, LRR'16]	$O(1)$ when $u_{i,j} \in \{\frac{-1}{\sqrt{n}}, 0, \frac{+1}{\sqrt{n}}\}$	$O(1)$
Kómlos conj.	$O(1)$	

$(m \gg n)$

	$x \in \{\pm 1\}^n$	$\|x\|_2 = \sqrt{n}$
Random x	$O(\sqrt{\log m})$	$O(\sqrt{\log m})$
Partial coloring	$O(\sqrt{\log \frac{m}{n}})$ when $u_{i,j} \in \{\frac{-1}{\sqrt{n}}, 0, \frac{+1}{\sqrt{n}}\}$	$O(\sqrt{\log \frac{m}{n}})$
This work		$\sqrt{2 \ln \frac{m}{n}}(1+\varepsilon)$

Our Algorithm

Theorem

Given unit vectors $u_1, \ldots, u_m \in \mathbb{R}^n$ and $m \geq 16n$, there is a poly(m, n) deterministic alg. outputting a vector x with $\|x\|_2 = \sqrt{n}$ satisfying, for all i,

$$\langle u_i, x\rangle \leq \sqrt{2 \ln \frac{m}{n}} \cdot \left(1 + O\left(\frac{1}{\log \frac{m}{n}}\right)\right)$$

Compare to $O\left(\sqrt{\log \frac{m}{n}}\right)$ from the partial coloring technique.

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Compare to $O\left(\sqrt{\log \frac{m}{n}}\right)$ from the partial coloring technique.
This is nearly the optimal bound in the "large cap regime":

Theorem [Böröczky-Winters '03]

For every choice of $2^{-o(\sqrt{n})} < \delta < \frac{1}{n^2}$, there is a set of $m \approx 1/\delta$ unit vectors $u_1, \ldots, u_m \in S^n$ such that, for any x with $\|x_2\| = \sqrt{n}$ there is a u_i with

$$\langle u_i, x \rangle \geq \sqrt{2 \ln \frac{m}{n}} \cdot \left(1 - O\left(\frac{1}{\sqrt{\log \frac{m}{n}}}\right)\right)$$

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Our Algorithm

Technique: adapt a deterministic **M**ultiplicative **W**eight **U**pdate algorithm for partial coloring due to [Levy-Ramadas-Rothvoss '18]

MWU for Spherical Discrepancy

Input: unit vectors $u_1, \dots, u_m \in S^n$

$x \leftarrow 0^n$

$w_i \leftarrow \exp(-\lambda^2)$

for $t = 1, \dots, T$:

- $I \leftarrow \{i : w_i \geq 2\}$
- $P \leftarrow \{x\} \cup \{\sum_i w_i u_i\} \cup \{u_i : i \in I\}$
- $M \leftarrow \sum_{i \notin I} w_i u_i u_i^\intercal$
- $y \leftarrow$ minimizer of $y^\intercal M y$ among $S^n \cap P^\perp$
- $x \leftarrow x + \delta y$
- **for** $i = 1, \dots, m$:
  - $w_i \leftarrow w_i \cdot \exp(\lambda \langle u_i, \delta y \rangle) \cdot \rho$

return x

Parameters:

$$\lambda = \sqrt{\ln \frac{m}{n}}, \quad \delta = \frac{1}{n^3}, \quad T = \frac{2n}{\delta^2}, \quad \rho = \exp\left(\frac{-\delta^2\lambda^2}{2n} \cdot (1 + \lambda\delta n)\right)$$

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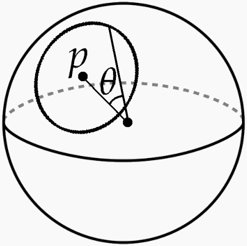
A Geometric Question

A *spherical cap* is a set of the form $\{x \in S^n : \langle x, p\rangle \geq \cos\theta\}$.

The *fractional size* of a spherical cap C is $\delta = \mathrm{vol}(C)/\,\mathrm{vol}(S^n)$.



Fix a parameter $\delta = \delta(n) \in (0, 1/2)$. How many spherical caps of fractional size δ are required to cover S^n?

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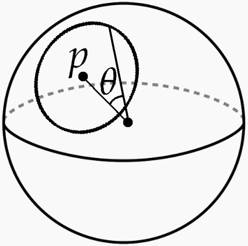

Fix a parameter $\delta = \delta(n) \in (0, 1/2)$. How many spherical caps of fractional size δ are required to cover S^n?

Call $m = m(n, \delta)$ the minimum number.

Trivial volume bound: need at least $m \cdot \delta \geq 1$. This could only be obtained if the caps could be perfectly disjoint, which is impossible.

Conjecture

$m \cdot \delta \geq \Omega(n)$

True if $\delta = \Omega(1)$ [Lusternik-Schnirelman theorem]
True if $\delta \leq n^{-n/2+o(n)}$ [Coxeter-Few-Rogers '59]

Algorithmic Lower Bounds

Fix a parameter $\delta = \delta(n) \in (0, 1/2)$. How many spherical caps of fractional size δ are required to cover S^n?

Theorem

If m spherical caps of size $2^{-o(\sqrt{n})} \leq \delta < 1/2$ cover S^n, then

$$m \cdot \delta \geq \Omega\left(\frac{n}{\sqrt{\log \frac{m}{n}}}\right)$$

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Proof idea: Given a small set of poles $u_1, \ldots u_m \in S^n$, run the Spherical Discrepancy algorithm to produce $x \in S^n$ which is far away from the u_i,

$$\langle u_i, x\rangle < \sqrt{\frac{2 \ln \frac{m}{n}}{n}} \cdot \left(1 + O\left(\frac{1}{\log \frac{m}{n}}\right)\right)$$

x is not in caps $\{y \in S^n : \langle y, u_i\rangle \geq \text{RHS}\}$.
Solve for $\cos\theta = \text{RHS}$ and then for δ corresponding to θ.

Gaussian Covering Problem

A *halfspace* is a set of the form $\{x \in \mathbb{R}^n : \langle x, u \rangle \geq L\}$ where $u \in S^n$.

The *Gaussian measure* of a halfspace H is $\Pr_{X \sim N(0,I)}[X \in H]$.

Fix a parameter $\delta = \delta(n)$. How many halfspaces of Gaussian measure δ are required to 1/2 cover an n-dimensional Gaussian random variable?

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Trivial volume bound: $m \cdot \delta \geq 1/2$.

Theorem

If m halfspaces of Gaussian measure $\delta < \frac{1}{2}$ cover the $\sqrt{n}$-sphere in $\mathbb{R}^n$, then

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x is not in halfspaces $\{y \in \mathbb{R}^n : \langle y, u_i\rangle \geq \text{RHS}\}$ which have Gaussian measure

$$\delta \approx \exp(-(\text{RHS})^2/2)/\text{RHS} = \Omega\left(\frac{n}{m \cdot \sqrt{\log \frac{m}{n}}}\right)$$

When is a halfspace approximately a cap?

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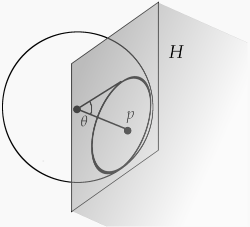
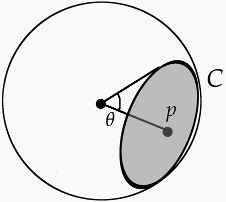


$$H = \{x \in \mathbb{R}^n : \langle x, p\rangle \geq \sqrt{n}\cos\theta\}$$

$\approx$



$$C = \{x \in S^n : \langle x, p\rangle \geq \cos\theta\}$$

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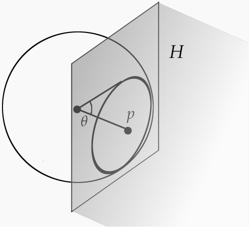

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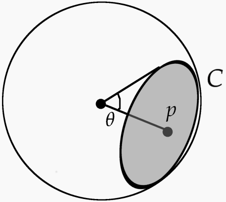

$$H = \{x \in \mathbb{R}^n : \langle x, p\rangle \geq \sqrt{n}\cos\theta\}$$

$$C = \{x \in S^n : \langle x, p\rangle \geq \cos\theta\}$$

Lemma (large cap regime)

As long as $\text{vol}(C) \geq 2^{-o(\sqrt{n})}$, then $\text{vol}(C) \sim \gamma(H)$.

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Hypercube $\to$ Hypersphere Relaxation

Strategy to solve a problem $\max_{x \in \{\pm 1\}^n} f(x)$:
(1) Relax the problem to $\max_{x \in \sqrt{n}\text{-sphere}} f(x)$
(2) Optimally solve the relaxed problem.
(3) Smartly round the solution to $\{-1, +1\}^n$.

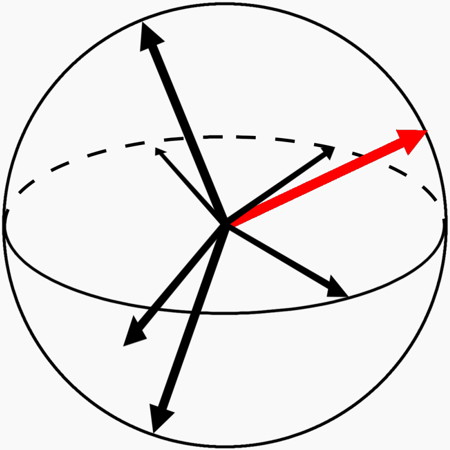
Question

Are there non-degree-2 f which can be attacked using this strategy?

Other problems:

- For $\delta < 2^{-\sqrt{n}}$, we have no nontrivial lower bound! Does the algorithm achieve anything here or is it inherently "Gaussian"?
- Chromatic number of the spherical distance graph $\{(x, y) \in (S^n, S^n) : \langle x, y \rangle \leq \cos\theta\}$.
- Brownian motion algorithm?
- Improve the $(1 + \varepsilon)$ error term to remove the $\sqrt{\log \frac{m}{n}}$ factor.

Thank you!



Thank you!

Open Problem #1: Improve the Error Bounds

Theorem

Given unit vectors $u_1, \ldots, u_m \in \mathbb{R}^n$ and $m \geq 16n$, there is a poly(m, n) deterministic algorithm that outputs a vector x satisfying

$$\langle u_i, x \rangle \leq \sqrt{\frac{2 \ln \frac{m}{n}}{n}} \cdot \left(1 + O\left(\frac{1}{\log \frac{m}{n}}\right)\right)$$

Conjecture

The inner product can be improved to

$$\langle u_i, x \rangle \leq \sqrt{\frac{2 \ln \frac{m}{n}}{n}} \cdot \left(1 - \frac{\ln \ln \frac{m}{n}}{4 \ln \frac{m}{n}} + O\left(\frac{1}{\log \frac{m}{n}}\right)\right)$$

This would remove the log factor from the $\Omega\left(\frac{n}{\sqrt{\log(1+\frac{m}{n})}}\right)$ covering density lower bound.

For $\delta < 2^{-\sqrt{n}}$, we had no bounds at all! Can we improve the analysis of the MWU algorithm to get a nontrivial bound here?

Open Problem #2: Nonlinear Functions

A spherical cap is a halfspace in S^n. Can we strengthen the analysis to sets that are slightly nonlinear?

Define a distance graph in spherical space $S^n_{\geq\theta}$ by taking edges between points at angle at least θ.

An independent set in this graph is a set with diameter $\leq \theta$.

The chromatic number of this graph is finite; reasonable color classes are spherical caps of angular diameter θ.

Conjecture

For every θ, $\chi(S^n_{\geq\theta}) \geq \Omega(m_{n,\theta})$ where $m_{n,\theta}$ is the minimum number of spheres of diameter θ needed to cover S^n.

Open Problem #3: Hypercube$\to$ Hypersphere Relaxation

Strategy to solve a problem $\max_{x \in \{\pm 1\}^n} f(x)$:

(1) Relax the problem to $\max_{x \in \sqrt{n}\text{-sphere}} f(x)$
(2) Optimally solve the relaxed problem.
(3) Smartly round the solution to $\{-1, +1\}^n$.

Question

What combinatorial optimization problems can be solved by relaxing $\{+1, -1\}^n$ to the $\sqrt{n}$-sphere?

Spectral algorithms apply this strategy to $f =$ degree-2 polynomial. Can any non-degree-2 polynomial be solved this way?

Hardness of Approximation

Spherical Discrepancy Problem

Given unit vectors $u_1, \ldots, u_m \in S^n$, compute

$$\min_{x \in S^n} \max_i \langle u_i, x \rangle .$$

A C-approximation is an algorithm which on an instance with value OPT outputs an x with value at most $C \cdot \mathrm{OPT}$.

Is it possible that a polynomial time $(1+\varepsilon)$-approximation exists for any fixed desired accuracy ε?

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Theorem

There is a constant $C > 1$ so that it is NP-hard to C-approximate Spherical Discrepancy.

Proof idea: Gap-preserving gadget reduction from MAX NAE-3-SAT.